§4.5 Isolated Singular Points

Def 1: A singular point z_0 is said to be isolated if there is a deleted ε-neighborhood $0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic.

Note: Then by Laurent series expansion about z_0, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

(where $a_n = c_n$, $b_n = c_{-n}$)

Def 2: (1) The pation

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

is called the principal part of f at z_0.

(2) Removable singular points

If $b_1 = b_2 = \cdots = b_n = \cdots = 0$, then z_0 is called a removable singular point of f.

(3) Essential Singular points

If $\exists$ infinitely many $b_n \neq 0$, then z_0 is called an essential singular point of f.

(4) Poles of order m

If $\exists\ m \geq 1$ such that $b_m \neq 0$, but

$$b_{m+1} = b_{m+2} = \cdots = 0,$$

then z_0 is called a pole of order m of f. And a pole of order $m=1$ is called a simple pole.

Notes: (1) z_0 = removable singular point of f

$$\Rightarrow \quad f(z) = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$
$$0 < |z-z_0| < R_1 \quad (\text{for some } R_1 > 0)$$

and can be extended to an analytic function

$$f(z) = \begin{cases} f(z), & 0 < |z-z_0| < R_1 \\ a_0, & \text{if } z = z_0 \end{cases} \quad \text{on } \{|z-z_0| < R_1\}$$

(2) z_0 = pole of order m of f

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m} \quad \begin{pmatrix} b_m \neq 0 \\ 0 < |z-z_0| < R_1 \end{pmatrix}$$

$$\Rightarrow (z-z_0)^m f(z) = b_m + \cdots + b_1 (z-z_0)^{m-1} + a_0 (z-z_0)^m + \cdots + a_n (z-z_0)^{m+n} + \cdots$$
$$(0 < |z-z_0| < R_1)$$

has a removable singular point at z_0 & hence by note (1), it can be extended to an analytic function in $\{|z-z_0| < R_1\}$.

Thm 1 : Let z_0 be an isolated singular point of a function f.

Then the followings are equivalent :

(a) z_0 is a pole of order m of f,

(b) $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and nonzero at z_0.

Pf (a) $\Rightarrow$ (b) is clear from note 2 with

$$\phi(z) = \begin{cases} b_m + \cdots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots, & 0<|z-z_0|<R_1 \\ b_m \; (\neq 0), & z=z_0. \end{cases}$$

(b) $\Rightarrow$ (a). If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, ϕ analytic & $\phi(z_0) \neq 0$,

Taylor's expansion $\Rightarrow$ $\phi(z) = a_0 + a_1(z-z_0) + \cdots$

with $a_0 = \phi(z_0) \neq 0$.

$$\Rightarrow f(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots \quad \text{with } a_0 \neq 0$$

$\therefore$ z_0 is a pole of order m. ※

Furthermore, we have the following

Thm 2 : Suppose that

(a) $p(z)$ & $q(z)$ are analytic at a point z_0;

(b) $p(z)$ and $q(z)$ has a zero of order n and m, respectively, at z_0.

Then $f(z) = \dfrac{p(z)}{q(z)}$ has a

$$\begin{cases} \text{removable singular point}, & \text{if } n = m \\ \text{zero of order } n-m, & \text{if } n > m \\ \text{pole of order } m-n, & \text{if } m > n \end{cases} \quad \text{at } z_0.$$

Pf: By Thm 1 of §4.4, $p(z) = (z-z_0)^n g(z)$

$q(z) = (z-z_0)^m h(z)$

with g, h analytic & $g(z_0) \neq 0$, $h(z_0) \neq 0$.

Therefore $f(z) = (z-z_0)^{n-m} \phi(z)$

where $\phi(z) = \dfrac{g(z)}{h(z)}$ analytic at z_0 and $\phi(z_0) \neq 0$.

The conclusion follows easily. ※

egs: (i) $\dfrac{1-\cosh z}{z^2} = -\dfrac{1}{2} - \dfrac{z^2}{4!} - \cdots$ $z=0$ removable.

& $\begin{cases} \dfrac{1-\cosh z}{z^2}, & z \neq 0 \\ -\dfrac{1}{2}, & z=0 \end{cases}$ is analytic at $z=0$.

(ii) $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots + \frac{1}{n!}\frac{1}{z^n} + \cdots \quad (0<|z|<\infty)$

$\therefore\ z=0$ is an essential singular point.

(iii) $\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad (0<|z|<1)$

$\therefore\ z=0$ is a pole of order 2

Behavior of the functions near isolated singular points

(a) Removable Singular Points

Thm 3: If z_0 is a removable singular point of f, then f is bounded and analytic in $0<|z-z_0|<\varepsilon$ for some $\varepsilon>0$.

Thm 4 (Riemann's Thm)

Suppose that f is bounded and analytic in $0<|z-z_0|<\varepsilon$ for some $\varepsilon>0$. Then either f is analytic at z_0 or f has a removable singular point at z_0.

(b) Essential Singular Point

Thm 5 (Casorati-Weierstrass Thm)

Suppose that z_0 is an essential singular point of f, and w_0 be any complex number. Then $\forall\ \varepsilon>0$, and $\forall\ \delta>0$,

$\exists\ z \in \{0 < |z - z_0| < \delta\}$ such that

$$|f(z) - w_0| < \varepsilon.$$

Remark: This implies that if z_0 is an essential singular point
(Ex) of f, then $\forall\, w_0 \in \mathbb{C}$, $\exists\, z_n \to z_0$ with $z_n \neq z_0$
such that $f(z_n) \to w_0$ as $n \to \infty$.

(c) Pole of order m

Thm6 If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$.

Proof of Theorems:

- Proof of Thm3 : Ex!
- Proof of Thm6 = Ex!
- Proof of Thm4 (Riemann's Thm)

f analytic in $0 < |z - z_0| < \varepsilon$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where $b_n = c_{-n} = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(s)}{(s - z_0)^{-n+1}}\, ds$,

(C: can be chosen as positive oriented $|z - z_0| = \rho < \varepsilon$)
(any ρ)

Since f is bdd in $0<|z-z_0|<\varepsilon$,

$\exists$ $M>0$ such that $|f(z)|\leq M$ $\forall$ $0<|z-z_0|<\varepsilon$.

$$\Rightarrow \quad |b_n| \leq \frac{1}{2\pi}\cdot\frac{M}{\rho^{-n+1}}\cdot 2\pi\rho = M\rho^n \qquad \forall\, n=1,2,3,\cdots \text{ and } 0<\rho<\varepsilon.$$

Letting $\rho\to 0$, we have $b_n=0$, $\forall$ $n=1,2,3,\cdots$.

$\therefore$ z_0 is a removable singular point. #

Proof of Thm 5

(Consider sufficiently small $\delta>0$ s.t. $f(z)$ is analytic in $0<|z-z_0|<\delta$.)

Suppose that the thm is not true, $\exists$ $\varepsilon>0$ and $\exists$ sufficiently small $\delta>0$ s.t.

$$|f(z)-w_0|\geq\varepsilon, \quad \forall\, 0<|z-z_0|<\delta$$

Then $g(z)=\dfrac{1}{f(z)-w_0}$ is bdd and analytic in $0<|z-z_0|<\delta$.

Hence z_0 is a removable singular point of $g(z)$ by Riemann's Thm (Thm 4).

If $g(z_0)\neq 0$, then $g(z)\neq 0$ in a nbd. of z_0 and

$f(z)=\dfrac{1}{g(z)}+w_0$ is analytic in a nbd. of z_0

$\Rightarrow$ z_0 is removable which is a contradiction.

If $g(z_0)=0$, then $g(z) = (z-z_0)^m h(z)$ in $0<|z-z_0|<\delta$

for some $m \geq 1$ and analytic $h(z)$ with $h(z_0) \neq 0$.

(Otherwise $g(z) \equiv 0 \Rightarrow f(z) = \infty$ in $0<|z-z_0|<\delta_0$ Contradiction)

$$\Rightarrow f(z) = \frac{1}{(z-z_0)^m h(z)} + w_0 \quad \text{in } 0<|z-z_0|<\delta$$

$\Rightarrow$ z_0 is a pole of order m of f.

Contradiction again.

This completes the proof of the theorem. ✗

Ch 5 Residue Theory and Applications

§5.1 Residue

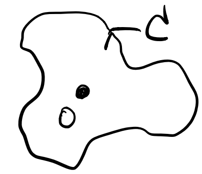
<u>Def 1</u> : The <u>residue</u> of f at an isolated singular point z_0 is the coefficient $b_1 = c_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$ of the term $\frac{1}{z-z_0}$ in the Laurent expansion of f about z_0, and is denoted by

$$\operatorname{Res}_{z=z_0} f(z) = b_1 \quad (= c_{-1})$$

(where C is any positively oriented simple closed contour surrounding z_0 & lies inside $0<|z-z_0|<\varepsilon$)

<u>eg 1</u> : $f(z) = \dfrac{e^z - 1}{z^5} = \dfrac{1}{z^4} + \cdots + \dfrac{1}{4!}\dfrac{1}{z} + \cdots \quad (0<|z|<\infty)$

$\therefore \operatorname{Res}_{z=0} f(z) = \frac{1}{4!} = \frac{1}{24}$



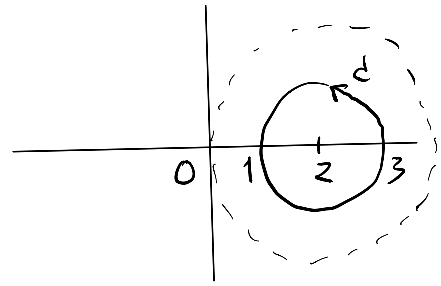
Hence $\int_C f(z)dz = 2\pi i \cdot \frac{1}{24} = \frac{\pi i}{12}$

for any +ve oriented simple closed contour C surrounding $z=0$

<u>eg 2</u> : $\cosh\left(\frac{1}{z^2}\right) = 1 + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{4!}\frac{1}{z^8} + \cdots$

$\Rightarrow \operatorname{Res}_{z=0} \cosh\left(\frac{1}{z^2}\right) = 0 \Rightarrow \int_C \cosh\left(\frac{1}{z^2}\right)dz = 0 \quad \forall C$ as in eg 1.

eg3: $\dfrac{1}{z(z-2)^5}$



$$\int_C \frac{dz}{z(z-2)^5} = 2\pi i \operatorname{Res}_{z=2} \frac{1}{z(z-2)^5}$$

Note $\dfrac{1}{z(z-2)^5} = \dfrac{1}{2(z-2)^5} + \cdots + \dfrac{1}{2^5}\cdot\dfrac{1}{(z-2)} + \cdots$ (Ex)

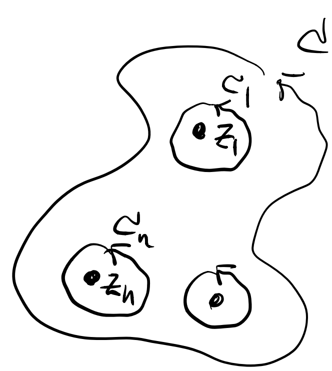
$\therefore \operatorname{Res}_{z=2} \dfrac{1}{z(z-2)^5} = \dfrac{1}{32}$.

$\therefore \displaystyle\int_C \frac{dz}{z(z-2)^5} = \frac{\pi i}{16}$.

Cauchy's Residue Theorem

Thm 1 Let C be a positively oriented simple closed contour. If f is analytic inside and on C except finitely many singular points $z_1, \dots, z_n$ inside C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)}$$



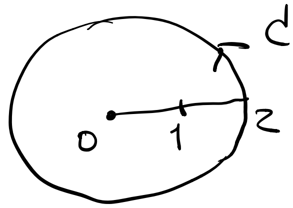
Pf: Let C_k be positively oriented circle centered at z_k with sufficiently small radius such that C_k interior to C and $\{C_k\}$ are disjoint.

Then Cauchy-Goursat Thm $\Rightarrow$

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z).$$

$\#$

eg $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$



$$\| $$

$$2\pi i\left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z)\right) \qquad \text{where } f(z) = \frac{4z-5}{z(z-1)}$$

$$= 2\pi i\,(5-1) = 8\pi i \quad (\text{Ex!})$$

$$\left(\text{Note: } \frac{4z-5}{z(z-1)} = \frac{5}{z} + \frac{(-1)}{z-1}\right)$$

Residue at Infinity

Suppose $f(z)$ is analytic in $R_1 < |z| < \infty$

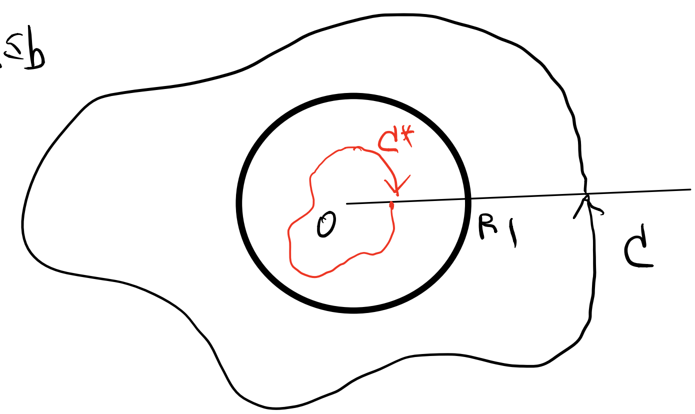
and $C = z(t)$, $a \le t \le b$

simple closed contour

in $R_1 < |z| < \infty$

surrounding $z = 0$

(+ve oriented)



Then $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

Change of variable $\zeta = \frac{1}{z}$.

Then $C^* = \zeta(t) = \frac{1}{z(t)}$ is a negatively oriented simple closed contour surrounding $z=0$.

Note that $\zeta'(t) = -\frac{z'(t)}{z^2(t)} \Rightarrow z'(t) = -\frac{\zeta'(t)}{\zeta^2(t)}$.

$$\therefore \int_C f(z)dz = \int_a^b f\left(\frac{1}{\zeta(t)}\right)\left(-\frac{\zeta'(t)}{\zeta^2(t)}\right) dt$$

$$= -\int_a^b \left[\frac{1}{\zeta^2(t)} f\left(\frac{1}{\zeta(t)}\right)\right] \zeta'(t)dt$$

$$= -\int_{C^*} g(\zeta)\, d\zeta \quad \text{where } g(\zeta) = \frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)$$

$$= \int_{-C^*} g(\zeta) d\zeta$$

$$= 2\pi i \operatorname*{Res}_{\zeta=0} \left[\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)\right]$$

Since f has no singular point in $R_1 < |z| < \infty$, $\zeta = 0$ is the only singular point of $g(\zeta) = \frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)$ inside $-C^*$.

In particular, we have

Thm2 If f is analytic everywhere in $\mathbb{C}$ except finitely many isolated singular points interior to a positively oriented simple closed contour C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]}$$

(Pf: Already proved!)

eg $\displaystyle\int_C \underbrace{\frac{z^3(1-3z)}{(1+z)(1+2z^4)}}_{f(z)}\,dz$, $C: |z|=3$, +ve oriented.

no singular pt. outside

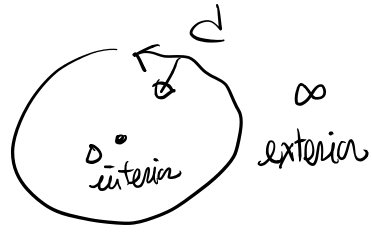
$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\,\frac{\left(\frac{1}{z}\right)^3\left(1-3\left(\frac{1}{z}\right)\right)}{\left(1+\frac{1}{z}\right)\left(1+2\left(\frac{1}{z}\right)^4\right)}$$

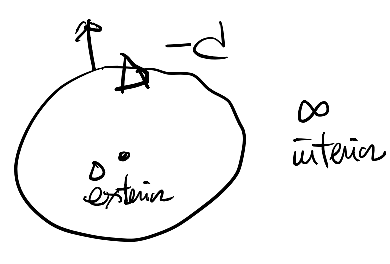
$$= \frac{z-3}{z(1+z)(2+z^4)} = -\frac{3}{2}\cdot\frac{1}{z} + \cdots$$

$$\therefore \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = -\frac{3}{2}$$

Hence $\displaystyle\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = -3\pi i$. #

Note:





Therefore, we may think that $z=\infty$ is interior to $-C$

$$\therefore \quad \frac{1}{2\pi i}\int_{-C} f(z)\,dz = -\operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

can be defined as the residue of f at $z=\infty$.

Then

$$\boxed{\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]}$$

Residues at Poles

Thm 3 : Let z_0 be a pole of order $m\ (\geq 1)$ of f and

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \quad 0<|z-z_0|<R \quad (\text{for some } R>0)$$

with $\phi(z)$ analytic in $|z-z_0|<R$ & $\phi(z_0)\neq 0$.

Then

$$\boxed{\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$

(In particular, $\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m=1$, ie simple pole)

Pf: Let $\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\Rightarrow \quad f(z) = \frac{a_0}{(z-z_0)^m} + \cdots + \frac{a_{m-1}}{(z-z_0)} + a_m + \cdots$$

$\therefore \quad \operatorname{Res}_{z=z_0} f(z) = a_{m-1} = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ ※

Thm 4: Let $p(z)$, $q(z)$ be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ & $q'(z_0) \neq 0$. Then z_0 is a simple pole of $f(z) = \dfrac{p(z)}{q(z)}$ with

$$\boxed{\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

Pf: Write $q(z) = (z - z_0)g(z)$ with $g(z)$ analytic and $g(z_0) \neq 0$. In fact, $q'(z) = g(z) + (z - z_0)g'(z)$

$\Rightarrow \quad q'(z_0) = g(z_0)$.

By Thm 3, $\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}_{z=z_0} \dfrac{p(z)}{(z-z_0)g(z)}$

$$= \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)} \quad ※$$

eg: $\operatorname{Res}_{z=n\pi} \cot z = \operatorname{Res}_{z=n\pi} \dfrac{\cos z}{\sin z}$

$\cos n\pi = (-1)^n \neq 0$,

$\sin n\pi = 0$, $(\sin z)'_{z=n\pi} = \cos n\pi = (-1)^n \neq 0$

$\therefore$ Thm 4 $\Rightarrow$ $\operatorname{Res}_{z=n\pi} \cot z = \dfrac{\cos n\pi}{\cos n\pi} = 1$. ※

§5.2 Application to (Real) Improper Integrals

Recall: (i) $\int_0^{\infty} f(x)dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_0^R f(x)dx$ (if exists)

(ii) $\int_{-\infty}^{\infty} f(x)dx \overset{\text{def}}{=} \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_0^{R_1} f(x)dx$

(if both limits exist)

(iii) Cauchy Principal Value (P.V.)

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \quad \text{(if exists)}$$

Note: • $\int_{-\infty}^{\infty} f(x)dx$ exists (as in (ii))

$$\begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \text{P.V.} \int_{-\infty}^{\infty} f(x)dx \text{ exists}$$

• However, if f is an even function ($f(-x) = f(x)$),

then $2\int_0^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx$.