## §2.14 Analytic (or holomorphic) functions

<u>Def</u>: (1) A function $f(z)$ is <u>analytic in an open set</u> $S$ if $f'(z)$ exists $\forall z \in S$.

(2) A function $f$ said to be <u>analytic</u> in a set $S$ that is <u>not open</u> if $f$ is analytic in some open set $S'$ containing $S$. In particular, a function $f(z)$ is <u>analytic at a point</u> $z_0$ if $f(z)$ analytic in $|z-z_0|<\varepsilon$ for some $\varepsilon>0$.

<u>Remark</u>: "Analytic at a point" is a stronger condition than "(cpx) differentiable at a point". But "analytic in an open set" $\Leftarrow\Rightarrow$ "(cpx) differentiable in an open".

<u>eg</u>: $f(z)=|z|^2$ is (cpx) differentiable at $z_0=0$, but not (cpx) differentiable for $z\neq 0$. Hence $f(z)=|z|^2$ is not analytic at $z_0=0$. (Ex!)

<u>Simple properties</u>:

connected open set in $\mathbb{C}$

(i) $f$ analytic in a domain $D \Rightarrow f$ continuous in $D$

(ii) analytic in $D \Rightarrow$ CR-eqts. in $D$

(iii) All $1^{st}$ order partial derivatives <u>exist and continuous</u> on $D$ and satisfy CR-eqts everywhere in $D$ $\Rightarrow$ analytic in $D$.

(iv) $f, g$ analytic $\Rightarrow$ $\begin{cases} f \pm g, \ fg \text{ analytic} \\ \dfrac{f}{g} \text{ analytic provided } g \neq 0 \end{cases}$

(In particular, rational function $\dfrac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$ is analytic in $\{z \in \mathbb{C} : b_0 + b_1 z + \cdots + b_m z^m \neq 0\}$)

(v) $f, g$ analytic $\Rightarrow$ $f \circ g$ analytic and

$$(f \circ g)'(z) = f'(g(z))\, g'(z).$$

<u>Thm</u>: If $f'(z) = 0$ everywhere in a domain $D$, then $f(z) = $ constant throughout $D$.

<u>Pf</u>: Let $f(z) = u + iv$, then

$$0 = f'(z) = u_x + i v_x \quad \text{(everywhere)}$$

$$\Rightarrow \quad u_x = v_x = 0$$

$$\text{CR-eqts} \Rightarrow \quad u_y = v_y = 0$$

$\therefore$ Advanced Calculus $\Rightarrow$ $\begin{cases} u \equiv u_0 \\ v \equiv v_0 \end{cases}$ constants.

$\Rightarrow \quad f(z) \equiv u_0 + i v_0$ . ※

<u>Cor 1</u>: If $f = u + iv$ and $\bar{f} = u - iv$ are both analytic in a domain $D$. Then $f \equiv$ constant on $D$.

<u>Pf</u>: $f$ analytic $\Rightarrow$ $f' = u_x + i v_x$ & $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$.

$\bar{f}$ analytic $\Rightarrow$ $\begin{cases} u_x = (-v)_y \\ u_y = -(-v)_x \end{cases}$ CR eqts for $\bar{f}$.

$$\Rightarrow \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$$

$$\Rightarrow \begin{cases} u_x = -u_x \Rightarrow u_x = 0 \\ u_y = -u_y \Rightarrow u_y = 0 \end{cases} \Rightarrow f' = 0$$

By Thm, $f \equiv$ constant. ※

<u>Cor 2</u>: If $f$ is analytic on a domain $D$ and $|f| \equiv$ const. on $D$, then $f \equiv$ const. on $D$.

<u>Pf</u>: Let $|f| \equiv r_0$ a real const. on $D$.

If $r_0 = 0$, then $f = 0$ on $D$. We've done.

If $r_0 \neq 0$, then $f(z) \neq 0$, $\forall z \in D$.

$$\Rightarrow \quad \bar{f}(z) = \frac{|f|^2}{f(z)} = \frac{r_0^2}{f(z)} \text{ is analytic in } D.$$

Ca 1 $\Rightarrow$ $f \equiv$ constant on $D$. #

## Ch3 Integrals

### §3.1 Derivatives and Integrals of Functions $w(t)$.

Def (1) $w(t) = u(t) + i v(t)$ is a cpx-valued function of a real variable $t \in (a,b)$. Then

$$\boxed{\frac{d}{dt} w(t) = w'(t) = u'(t) + i v'(t).}$$

(2) $w(t) = u(t) + i v(t)$ is a cpx-valued function of a real variable $t \in [a,b]$. Then

$$\boxed{\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt}$$

(provided $u', v', \int_a^b u, \int_a^b v$ exist.)

Fundamental Theorem of Calculus:

If $W(t) = U(t) + i V(t)$ and

$w(t) = u(t) + i v(t)$ are functions of $t \in [a,b]$

such that $W'(t) = w(t)$ $\forall t$. Then

$$\int_a^b w(t)\,dt = W(b) - W(a) \left( = [W(t)]_a^b = W(t)\Big|_a^b \right)$$

## §3.2 Contour Integrals

Def (1) Suppose that $C$ is a curve (contour) in $\mathbb{C}$. Then the contour integral of $f$ along $C$ is

$$\boxed{\int_C f(z)\,dz = \left(\int_C u\,dx - v\,dy\right) + i\left(\int_C v\,dx + u\,dy\right)}$$

for $f(z) = u(x,y) + i v(x,y)$.

(Alternately)
(2) If $C$ is parameterized by $z = z(t) = x(t) + i y(t)$,
then $\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt}$

Check: Def (1) $\Rightarrow$ Def (2)

$$\int_a^b f(z(t))\, z'(t)\,dt = \int_a^b [u(z(t)) + i\, v(z(t))][x'(t) + i y'(t)]\,dt$$

$$= \int_a^b [u(z(t))x'(t) - v(z(t))y'(t)]\,dt$$

$$+ i \int_a^b [v(z(t))x'(t) + u(z(t))y'(t)]\,dt$$

$$= \left(\int_C u\,dx - v\,dy\right) + i\left(\int_C v\,dx + u\,dy\right)$$

\#

<u>Note</u>: The value $\int_C f(z)\,dz$ is independent of the choice of the parameter $t$. (Ex!)

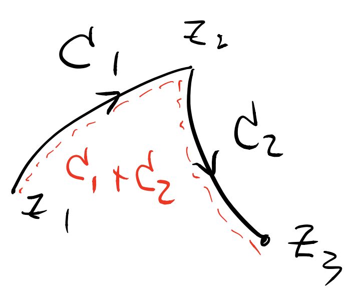
<u>Def</u> (1) Let $C$ be a contour represented by $z(t)$, $a \le t \le b$.

Then <u>$-C$</u> is the contour represented by

$$z = z(-t), \quad -b \le t \le -a.$$

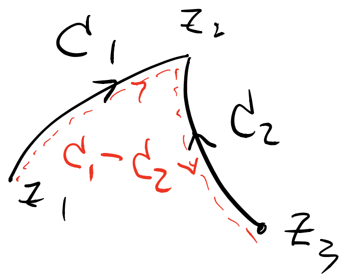
(2) If $C_1$ is a contour from $z_1$ to $z_2$, and $C_2$ is a contour from $z_2$ to $z_3$.

Then <u>sum</u> $C = C_1 + C_2$ is the contour from $z_1$ to $z_3$ by first travel from $z_1$ to $z_2$ along $C_1$ and then $z_2$ to $z_3$ along $C_2$.




(3) If $C_1$ is a contour from $z_1$ to $z_2$ and $C_2$ is a contour from $z_3$ to $z_2$.

Then $C_1 + (-C_2)$ is well-defined as in (2) and is denoted by $C_1 - C_2$.




<u>Properties</u> :

(1) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, for constant $z_0$.

(2) $\int_C [f(z) \pm g(z)]\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$

(3) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(4) $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

<u>Pf</u> : (1) & (2) are easy.

(3) Let $C : z = z(t)$, $a \le t \le b$.

Then $-C : z = z(-t)$, $-b \le t \le -a$.

$$\therefore \int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\left(\frac{d}{dt} z(-t)\right) dt$$

$$= \int_{-b}^{-a} f(z(-t))\left(-z'(-t)\right) dt$$

$$= -\int_a^b f(z(t))\, z'(t)\, dt$$
$$= -\int_C f(z)\, dz \, .$$

(4) Let $C_1 : z = z_1(t), \; a \le t \le c$
$\quad C_2 : z = z_2(t), \; c \le t \le b$ (by choosing suitable parameters)

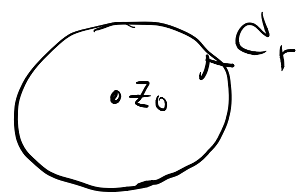
Then
$$C_1 + C_2 : z = \begin{cases} z_1(t), & a \le t \le c \\ z_2(t), & c \le t \le b \end{cases} .$$

$$\Rightarrow \int_{C_1+C_2} f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$
$$= \int_a^c f(z_1(t))\, z_1'(t)\, dt + \int_c^b f(z_2(t))\, z_2'(t)\, dt$$
$$= \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \quad \#$$

eg: Evaluate $\int_{C_1} \frac{dz}{z-z_0}$ along $C_r : |z - z_0| = r$

and in counterclockwise direction

Soln: Parametrize $C_r$ by




$$z = z(t) = z_0 + re^{it}$$
$$0 \le t \le 2\pi \, .$$

$$\Rightarrow \int_{C_1} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{z'(t)dt}{z(t)-z_0}$$

$$= \int_0^{2\pi} \frac{ire^{it}\,dt}{re^{it}}$$

$$= i\int_0^{2\pi} dt = 2\pi i \quad \#$$

## §3.3 Upper Bounds for Moduli of Contour Integrals

<u>Lemma</u>: If $w(t)$ is a piecewise continuous cpx-valued function defined on $a \le t \le b$, then

$$\left| \int_a^b w(t)dt \right| \le \int_a^b |w(t)|\,dt .$$

<u>Pf</u>: If $\int_a^b w(t)dt = 0$, then we are done.

If $\int_a^b w(t)dt \neq 0$, then it can be written as

$$\int_a^b w(t)dt = r_0 e^{i\theta_0},$$

where $r_0 = \left|\int_a^b w(t)dt\right| \neq 0$, $\theta_0 \in \mathbb{R}$.

Then $r_0 = e^{-i\theta_0}\int_a^b w(t)dt$

$$= \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$= \operatorname{Re}\left[\int_a^b e^{-i\theta_0} w(t)\,dt\right] \quad \text{since } r_0 \in \mathbb{R}$$

$$= \int_a^b \operatorname{Re}\left(e^{-i\theta_0} w(t)\right) dt$$

$$\leq \int_a^b \left|e^{-i\theta_0} w(t)\right| dt$$

$$= \int_a^b |w(t)|\,dt \quad \#$$

<u>Thm</u>: Let $C$ be a contour of length $L$, and $f(z)$ be a piecewise continuous function on $C$. Suppose $M \geqslant 0$ is a constant such that

$$|f(z)| \leq M, \quad \forall z \in C.$$

Then

$$\left|\int_C f(z)\,dz\right| \leq ML.$$

<u>Pf</u>: Parametrize $C$ by $z = z(t) = x(t) + iy(t)$, $a \leq t \leq b$.

Then
$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

By lemma
$$\left|\int_C f(z)\,dz\right| \leq \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\leq M \int_a^b |z'(t)|\,dt$$

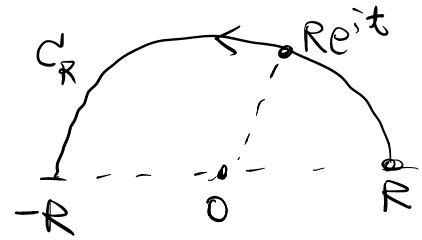
$$= M \int_a^b \sqrt{x'^2(t) + y'^2(t)}\, dt$$

$$= ML. \quad ※$$

eg: Let $C_R$ = semicircle $z = Re^{it}$, $0 \le t \le \pi$, for $R > 3$

Then

$$\lim_{R \to +\infty} \int_{C_R} \frac{(z+1)}{(z^2+4)(z^2+9)} dz = 0.$$




In fact, for $R > 3$, we have on $C_R$

$$\begin{cases} |z+1| \le |z| + 1 = R+1 \\ |z^2+4| \ge |z|^2 - 4 = R^2 - 4 \quad (>0) \\ |z^2+9| \ge |z|^2 - 9 = R^2 - 9 \quad (>0) \end{cases}$$

$$\Rightarrow \left| \frac{z+1}{(z^2+4)(z^2+9)} \right| \le \frac{R+1}{(R^2-4)(R^2-9)}, \quad \forall z \in C_R.$$

Note that the length of $C_R = \pi R$, we have

$$\left| \int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)} \right| \le \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R$$

$$= \frac{\pi R(R+1)}{(R^2-4)(R^2-9)} \to 0 \quad \text{as } R \to +\infty.$$

$$\therefore \quad \lim_{R\to+\infty} \int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)} = 0 \quad . \quad \#$$

## §3.4 Antiderivatives

Def: Let $f(z)$ be a cpx-valued continuous function in a domain $D$. Then the antiderivative of $f(z)$ on $D$ is a function $F(z)$ such that

$$F'(z) = f(z), \quad \forall z \in D.$$

Notes: (1) An antiderivative is an analytic function.

(2) An antiderivative of a given function is unique up to an additive constant.

( i.e. $F' = G' = f \Rightarrow F = G + \text{const.}$ )

Thm: Suppose that a function $f(z)$ is continuous in a domain $D$. Then the following statements are equivalent.

(a) $f(z)$ has an antiderivative $F(z)$ throughout $D$.

(b) $\forall$ contours $C_1$ and $C_2$ (in $D$) with the same initial and end points,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \quad \left(\begin{array}{l}\text{independent}\\ \text{of path}\end{array}\right)$$

(c) $\forall$ closed contour $C$ (in $D$)

$$\int_C f(z)\,dz = 0 .$$

If any of the above statement true, then the integral in (b)

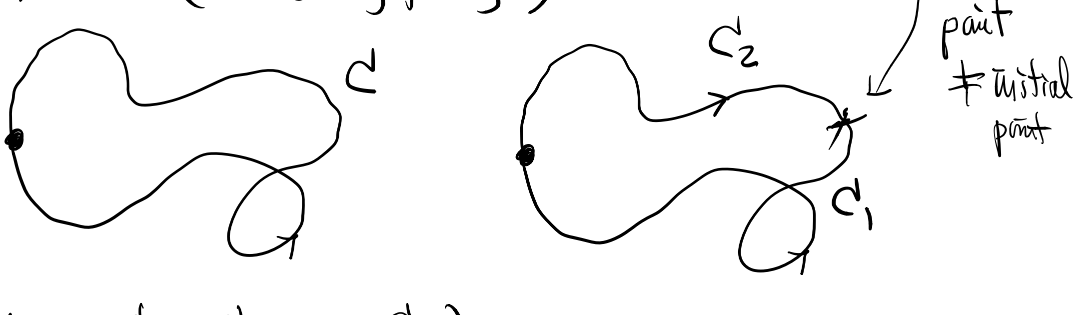
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F$ is the antiderivative given in (a), and $z_1, z_2$ are the common initial and end points of $C_1$ & $C_2$.

In this case, we denote

$$\int_{C_1} f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

Pf: (b) $\Rightarrow$ (c) (Sketch of proof)




Then $C = C_1 + (-C_2)$

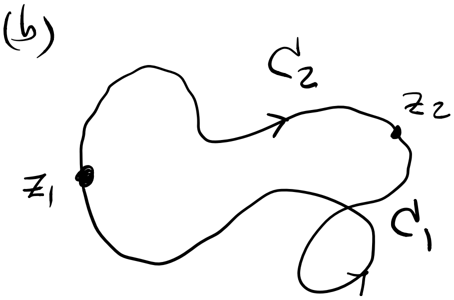
By (b)

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\Rightarrow \quad \int_{C} f(z)\,dz = \int_{C_1 + (-C_2)} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$$= 0 \,. \quad \#$$

(c) $\Rightarrow$ (b)




$\Rightarrow$ $C_1 + (-C_2)$ is a closed contour

By (c), $\displaystyle\int_{C_1 + (-C_2)} f(z)\,dz = 0$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \,. \quad \#$$

(a) $\Rightarrow$ (b).

If $C$ is a smooth arc from $z_1$ to $z_2$ & parametrized by $z = z(t)$, $a \le t \le b$. Then

$$\frac{d}{dt} F(z(t)) = F'(z(t))\,z'(t) = f(z(t))\,z'(t)$$

$$\therefore \int_{C} f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt$$

$$= F(z(b)) - F(z(a)) = F(z_2) - F(z_1) \,.$$

If $C$ is piecewise smooth: $C = C_1 + \cdots + C_N$ with $C_i$ smooth arc $\forall i$, joining $z_i$ to $z_{i+1}$.

Then $\int_{C_i} f(z)dz = F(z_{i+1}) - F(z_i)$, $\forall i$

$$\Rightarrow \int_C f(z)dz = \sum_{i=1}^{N} \int_{C_i} f(z)dz = \sum_{i=1}^{N} [F(z_{i+1}) - F(z_i)]$$

$$= F(z_{N+1}) - F(z_1)$$

$\therefore \int_C f(z)dz$ is independent of path, depends only on the initial & end points.

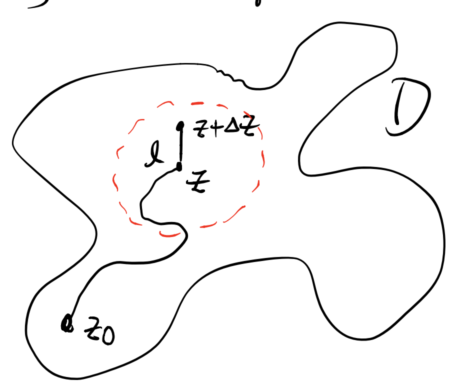
(And the required formula!) $\divideontimes$

(b) $\Rightarrow$ (a) Fix any $z_0 \in D$. Then for any $z \in D$, we define

$$F(z) = \int_{z_0}^{z} f(z)dz$$

which is well-defined because of the assumption (b).

For $|\Delta z|$ small, we can choose path as in the figure with $l$ is a straight line segment.




Then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

$$= \int_{z}^{z+\Delta z} f(s)\,ds$$

$$\Rightarrow \quad \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s)\,ds - f(z)$$

$$= \int_{z}^{z+\Delta z} \frac{f(s)-f(z)}{\Delta z}\,ds$$

( since $\int_{z}^{z+\Delta z} ds = \Delta z$ as $\frac{dz}{dz} = 1$ and we've proved (a) $\Rightarrow$ (b). )

Since $f$ is continuous, we have

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(s) - f(z)| < \varepsilon, \quad \forall\ |s - z| < \delta.$$

Therefore, for $|\Delta z| < \delta$ and evaluating the integral along the straight line segment $l$, we have

$$\left| \int_{z}^{z+\Delta z} \frac{f(s)-f(z)}{\Delta z}\,ds \right| \leq \frac{\varepsilon}{|\Delta z|} \cdot |\Delta z| = \varepsilon.$$

So we've proved that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right| < \varepsilon, \quad \forall \ |\Delta z| < \delta.$$

$$\therefore \quad F'(z) = f(z), \quad \forall z \in D. \ \#$$