Ch7 The 1^{st} & 2^{nd} variation formula

Let
- M = complete Riem mfd.
- $\gamma(t,u) : [a,b] \times [-\varepsilon, \varepsilon] \to M$ a C^∞ map
- $\{\gamma_u(t)\}$ corresponding 1-parameter family of curves with base curve γ_0 equal to a given curve $\gamma(t)$ parametrized by arc-length, i.e. $|\gamma'(t)| = 1$.
- U = transversal vector field of $\{\gamma_u\}$.
- T = tangent vector field along $\{\gamma_u\}$.

Then the length of $\gamma_u(t)$ is

$$L(u) = \int_a^b |\gamma_u'(t)|\, dt = \int_a^b |T|\, dt$$

$$\Rightarrow \quad \frac{dL}{du}(u) = \int_a^b \frac{d}{du} |T|\, dt$$

$$= \int_a^b U \sqrt{\langle T, T\rangle}\, dt$$

$$= \int_a^b \frac{\langle T, D_U T\rangle}{|T|}\, dt$$

$$= \int_a^b \frac{1}{|T|} \langle T, D_T U\rangle\, dt \qquad [T,U] = 0$$

Putting $u=0$,

$$\frac{dL}{du}(0) = \int_a^b \langle \gamma'(t), D_{\gamma'(t)} U \rangle \, dt$$

$$= \int_a^b \left[\frac{d}{dt} \langle \gamma'(t), U \rangle - \langle D_{\gamma'(t)} \gamma'(t), U \rangle \right] dt$$

where $U(t) = U(t, 0)$ is the transversal vector field along γ.

$$\boxed{\frac{dL}{du}(0) = \langle \gamma'(t), U(t) \rangle \Big|_a^b - \int_a^b \langle D_{\gamma'(t)} \gamma'(t), U(t) \rangle \, dt}$$

which is the <u>1st variation formula for arc-length</u>.

<u>Lemma 1</u>: A curve $\gamma : [a,b] \to M$ is a geodesic $\Leftrightarrow$ it is a critical point of the arc-length functional with respect to (all) <u>normal variations</u> $\{\gamma_u\}$ (i.e. $\forall u$, $\gamma_u(a) = \gamma(a)$ & $\gamma_u(b) = \gamma(b)$.)

<u>Pf</u>: For normal variations, $U(a) = U(b) = 0$

$$\therefore \quad \frac{dL}{du}(0) = -\int_a^b \langle D_{\gamma'} \gamma', U \rangle \, dt$$

$\forall\, U$ with $U(a) = U(b) = 0$.

$$\therefore \quad 0 = \frac{dL}{du}(0) \iff D_{\gamma'} \gamma' = 0 \quad (\text{Ex!})$$

$\forall\, U$ with $U(a) = U(b) = 0$

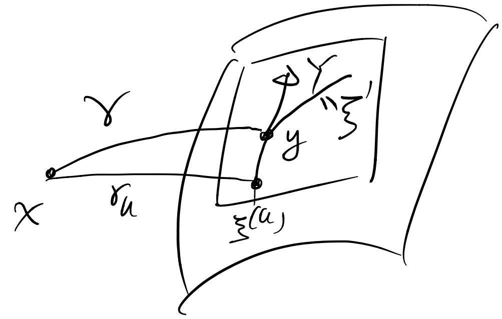
#

Lemma 2: Let
- N = closed submanifold of M
- $x \notin M$
- $y \in N$ such that $d(x,y) = d(x, N)$ $\overset{def}{=} \inf\{d(x,y) : y \in N\}$.
- $\gamma : [a,b] \to M$ shortest geodesic joining x to y.

Then γ is normal to N (i.e. $\underset{(\gamma(b)=y)}{\gamma'(b)} \perp T_yN$)

Pf: Let $Y \in T_yN$. We need to show that $\langle \gamma'(b), Y \rangle = 0$.

For this, take a C^∞ curve $\xi : [-\varepsilon, \varepsilon] \to N$ s.t. $\xi'(0) = Y$ $(\xi(0) = y)$



Let $\{\gamma_u\}$ be a 1-parameter family of curves given by

$$\gamma(t,u) : [a,b] \times [-\varepsilon, \varepsilon] \to M \quad \text{with}$$

$$\begin{cases} \gamma_0(t) = \gamma(t), & \forall\, t \in [a,b] \\ \gamma_u(a) = x, & \forall\, u \\ \gamma_u(b) = \xi(u) & \end{cases}$$

By assumption $L(0) = d(x,y) \le d(x, \xi(u)) \le L(u)$, $\forall\, u \in [-\varepsilon, \varepsilon]$

$$\Rightarrow \qquad \frac{dL}{du}(0) = 0 .$$

1^{st} variation formula $\Rightarrow$

$$0 = \langle \gamma'(t), U(t) \rangle \Big|_a^b - \int_a^b \langle \cancel{D_{\gamma'}\gamma'}^{\to 0}, U \rangle \, dt$$

$$= \langle \gamma'(b), U(b) \rangle - \langle \gamma'(a), U(a) \rangle$$

$$= \langle \gamma'(b), U(b) \rangle \qquad (\gamma_u(a) \equiv x, \ \forall u)$$

By $\gamma_u(b) = \xi(u)$, $\forall u$, we have

$$U(b) = \xi'(0) = Y .$$

$$\therefore \qquad 0 = \langle \gamma'(b), Y \rangle .$$

※

Now suppose that $\gamma : [a,b] \to M$ is a <u>normalized geodesic</u>.
We would like to calculate $\frac{d^2L}{du^2}(0)$ for the family $\{\gamma_u\}$.

We've proved that

$$\frac{dL}{du}(u) = \int_a^b \frac{1}{|T|} \langle T, D_T U \rangle \, dt$$

$$\Rightarrow \frac{d^2L}{du^2}(u) = \int_a^b \frac{\partial}{\partial u} \left[\frac{1}{|T|} \langle T, D_T U \rangle \right] dt$$

$$= \int_a^b \left[- \frac{1}{|T|^3} \langle T, D_T U \rangle^2 + \frac{1}{|T|} U \langle T, D_T U \rangle \right] dt$$

$$= \int_a^b \left[- \frac{1}{|T|^3} \langle T, D_T U \rangle^2 + \frac{1}{|T|} \langle D_U T, D_T U \rangle + \frac{1}{|T|} \langle T, D_U D_T U \rangle \right] dt$$

$$= \int_a^b \left\{ \begin{array}{l} -\frac{1}{|T|^3}\langle T, D_T U\rangle^2 + \frac{1}{|T|}|D_T U|^2 \\ \qquad + \frac{1}{|T|}\langle T, D_T D_U U + R_{TU}U\rangle \end{array} \right\} dt$$

(since $[U,T]=0$)

$$= \int_a^b \left\{ \begin{array}{l} -\frac{1}{|T|^3}\left[T\langle T, U\rangle - \langle D_T T, U\rangle\right]^2 + \frac{1}{|T|}|D_T U|^2 \\ \qquad + \frac{1}{|T|}\langle T, D_T D_U U\rangle - \frac{1}{|T|}\langle R_{UT}U, T\rangle \end{array} \right\} dt$$

Note that at $u=0$, $\left\{ \begin{array}{l} D_T T = D_{\gamma'}\gamma' = 0 \\ |T| = |\gamma'| = 1. \end{array} \right.$

$$\therefore \frac{d^2L}{du^2}(0) = \int_a^b \left\{ \begin{array}{l} -\left[\frac{d}{dt}\langle \gamma', U\rangle\right]^2 + |U'|^2 - \langle R_{U\gamma'}U, \gamma'\rangle \\ + \left(\frac{d}{dt}\langle \gamma', D_U U\rangle - \langle \cancel{D_{\gamma'}\gamma'}, D_U U\rangle\right) \end{array} \right\} dt$$

$\Rightarrow$

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_U U\rangle\Big|_a^b + \int_a^b \left\{\left[|U'|^2 - \left(\frac{d}{dt}\langle \gamma', U\rangle\right)^2\right] - \langle R_{U\gamma'}U, \gamma'\rangle\right\} dt}$$

which is the 2nd variation formula (for normalized geodesic)

Let $U^\perp = U - \langle U, \gamma'\rangle\gamma'$ the normal component of U, then the 2nd variation formula can be written as

$$\boxed{\frac{d^2L}{du^2}(0) = \langle \gamma', D_V V\rangle\Big|_a^b + \int_a^b \left\{ |D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp \gamma'} V^\perp, \gamma'\rangle \right\} dt}$$

Note: • If $\{\gamma_u\}$ is normal in the sense that

$$\gamma_u(a) = \gamma(a), \quad \gamma_u(b) = \gamma(b)$$

then $\langle \gamma', D_V V\rangle(a) = \langle \gamma', D_V V\rangle(b) = 0$

• If $\{\gamma_u\}$ is a 1-parameter of (smooth) closed curves, then $\langle \gamma', D_V V\rangle\Big|_a^b = 0$.

• The interior term

$$\int_a^b \left[|D_{\gamma'} V^\perp|^2 - \langle R_{V^\perp \gamma'} V^\perp, \gamma'\rangle \right] dt$$

is related to the Jacobi Operator on $V^\perp$ (under a suitable bdy condition).

Application 1

Thm3 Let • M = complete simply-connected Riem. mfd. with

• $K \le 0$ (sectional curvature)

• $O \in M$ is a fixed point.

• $\rho : M \to [0, \infty)$ (the distance function wrt O) is defined by $\rho(x) = d(x, O)$.

Then $\rho^2 \in C^\infty(M)$ and $D^2\rho^2 > 0$ (strictly positive definition)

Eg: If $M = \mathbb{R}^n$, O = origin, then $\rho^2(x) = |x|^2$ and

$$D^2\rho^2(v,v) = c|v|^2 \text{ (for some } c>0) \quad \text{(Ex!)}$$

Pf of Thm 3: By Cartan-Hadamard Thm,

$$\rho(x) = \left|(\exp_O)^{-1}(x)\right|$$

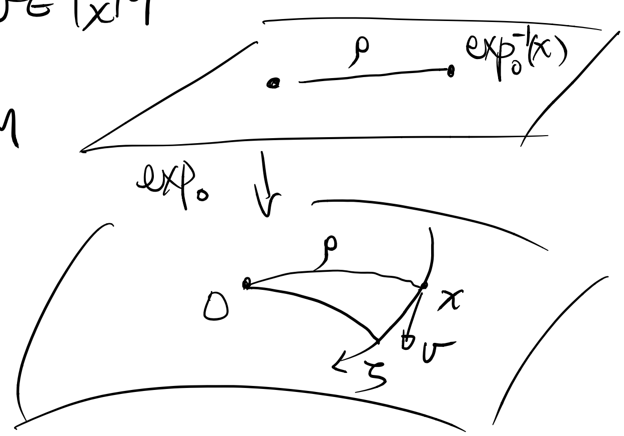
Therefore $\rho^2(x) = |(\exp_O)^{-1}(x)|^2$ is smooth.

Now suppose $x \neq O$, and $v \in T_xM$

Take a curve $\xi : [-\varepsilon, \varepsilon] \to M$

such that

$$\xi(0) = x \;\; \& \;\; \xi'(0) = v$$



For each $u \in [-\varepsilon, \varepsilon]$,

let $\gamma_u : [0,b] \to M$ (with $b = \rho(x)$, $a = 0$)

is the unique geodesic joining O to $\xi(u)$. Note that

$\gamma_0 = \gamma : [0,b] \to M$ is a normalized geodesic (other γ_u may not be normalized.)

Also, we can choose $\zeta(u)$ to be a geodesic. Then the end point of γ_u is $\gamma_u(b) = \zeta(u)$

$\Rightarrow$ the transversal vector field $V(t,u)$ at $t=b$ is $V(b,u) = \zeta'(u)$.

Therefore $$D_V V\Big|_{(b,u)} = D_{\zeta'(u)}\zeta'(u) = 0 \quad \left(\text{since } \zeta = \text{geodesic}\right)$$

On the other hand, $\gamma_u(0) = 0 \Rightarrow V(0,u) \equiv 0$

$$\Rightarrow D_V V\Big|_{(0,u)} = 0.$$

Hence, the 2^{nd} variation formula gives

$$\frac{d^2L}{du^2}(0) = \int_0^b \left\{ |D_{\gamma'}V^\perp|^2 - \langle R_{V^\perp \gamma'}V^\perp, \gamma'\rangle \right\} dt$$

$$\geq \int_0^b |D_{\gamma'}V^\perp|^2 \qquad (\text{since } K \leq 0)$$

Now $$D^2\rho^2(v,v) = \left\{ \zeta'(\zeta'\rho^2) - \cancelto{0}{(D_{\zeta'}\zeta')}\rho^2 \right\}\Big|_{u=0}$$

$$= \zeta'(\zeta'\rho^2)\Big|_{u=0} \qquad (\zeta = \text{geodesic})$$

$$= \zeta'(2\rho\,\zeta'\rho)\Big|_{u=0}$$

$$= \left[2\rho\,\zeta'(\zeta'\rho) + 2(\zeta'\rho)^2\right]_{u=0}$$

$$= 2\rho(x)\frac{d^2}{du^2}\Big|_{u=0}\rho(\zeta(u)) + 2\left[\frac{d}{du}\Big|_{u=0}\rho(\zeta(u))\right]^2$$

Note that $\rho(\zeta(u)) = L(\gamma_u) = L(u)$

$$\therefore \quad \left.\frac{d}{du}\right|_{u=0} \rho(\zeta(u)) = \frac{dL}{du}(0)$$

$$= \langle \gamma'(t), U(t) \rangle \Big|_0^b - \int_a^b \langle \cancel{D_{\gamma'}\gamma'}, U \rangle dt$$

(since γ = geodesic)

$$= \langle \gamma'(b), U(b) \rangle$$

$$= \langle \gamma'(b), \zeta'(0) \rangle$$

$$= \langle \gamma'(b), v \rangle$$

$$\& \quad \left.\frac{d^2}{du^2}\right|_{u=0} \rho(\zeta(u)) = \frac{d^2L}{du^2}(0) \geq \int_0^b |D_{\gamma'} U^{\perp}|^2 dt$$

$$\therefore \quad D^2\rho^2(v,v) \geq 2\rho(x) \int_0^b |D_{\gamma'} U^{\perp}|^2 dt + 2\left[\langle \gamma'(b), v \rangle\right]^2$$

If $\langle \gamma'(b), v \rangle \neq 0$, then $D^2\rho^2(v,v) > 0$.

If $\langle \gamma'(b), v \rangle = 0$, then $U(b) = v \in [\gamma'(b)]^{\perp}$.

Note that $\{\gamma_u\}$ is a 1-para. family of geodesics, U is a Jacobi field along γ. Hence

$$\langle \gamma'(b), U(b) \rangle = \langle \gamma'(0), U(0) \rangle = 0 \qquad (U(b) = v \neq 0)$$

$\Rightarrow$ $U(t)$ is <u>nontrivial</u> <u>normal</u> Jacobi field

$\therefore \quad U^{\perp}(t) = U(t)$

Therefore $D_{\gamma'} U^{\perp} = D_{\gamma'} U \not\equiv 0$. Otherwise, U is a parallel transport of $V(0)=0 \Rightarrow V\equiv 0$ which is a contradiction.

All together, we have $D^2\rho^2(v,v) \geq \int_0^b |D_{\gamma'}V^\perp|^2 dt > 0$ (for $v\neq 0$). This completes the proof of the thm. ※

The <u>key point</u> of the conclusion of the above thm is that $D^2\rho^2>0$ on the <u>whole</u> M, which needs the curvature assumption. Otherwise, we have

<u>Lemma 4</u> Let
- M = Riem mfd
- $O\in M$
- $\rho: M\to\mathbb{R}$ distance to O

Then $\exists$ a nbd $\mathcal{U}_0$ of O in M s.t. ρ^2 is smooth and $D^2\rho^2>0$ in $\mathcal{U}_0$.

<u>Sketch of Pf</u>: Let $\mathcal{U}$ be a nbd. of O s.t. $\exists$ normal coordinate system $\{x^1,\cdots,x^n\}$ centered at O. Using this, one can show that $v,w\in T_OM$,

$$D^2\rho^2(v,w) = 2\langle v,w\rangle \qquad (Ex!)$$

Therefore, at the center O, $D^2\rho^2>0$.

$\Rightarrow D^2\rho^2>0$ in a nbd $\mathcal{U}_0\subset\mathcal{U}$ of O. ※

Def: A function $f: M \to \mathbb{R}$ (M = Riem. mfd) is said to be convex (strictly convex)

$\iff$ $\forall$ geodesic γ in M, $f\circ\gamma$ is convex (strictly convex).

- Therefore, a C^∞ $f: M \to \mathbb{R}$ is convex (strictly convex)

  $\iff$ $D^2 f \geq 0$ (>0) (Ex!)

Def: Let M = complete Riem. mfd. Then

- a subset $\Omega \subset M$ is called convex

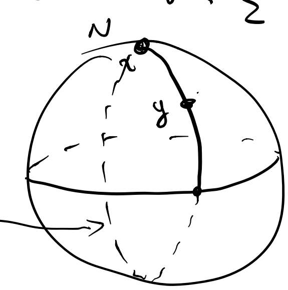
  $\iff$ $\forall x, y \in \Omega$, the shortest geodesic joining x to y is contained in Ω.

- a subset $\Omega \subset M$ is called totally convex

  $\iff$ $\forall x, y \in \Omega$, any geodesic joining x to y is contained in Ω.

Eg1: On $S^2 \subset \mathbb{R}^3$, geodesic ball of radius $r \leq \frac{\pi}{2}$ is convex, but not totally convex

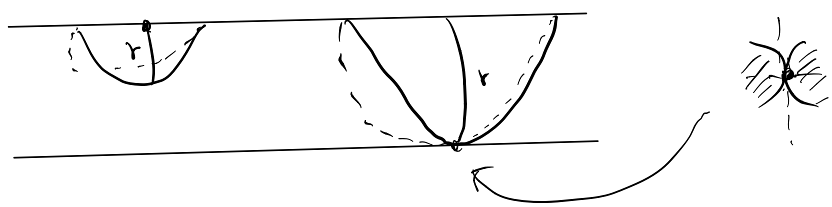
geodesic joining x to y not contained in $B_r(x)$



Furthermore, geodesic ball of radius r between $\frac{\pi}{2}$ & π is not even convex (Ex!)

Note: If M is a simply-connected complete Riem. mfd with nonpositive sectional curvature. Then Cartan-Hadamard $\Rightarrow$ any geodesic is minimizing. Therefore, a convex subset of M is also totally convex.

Eg 2 Cylinder $\{x^2+y^2=1\} \subset \mathbb{R}^3$. Then B_r is convex for $r \le \frac{\pi}{2}$, not convex for $r > \frac{\pi}{2}$:



Lemma 5 Let M = Riem mfd

(1) Let
- $\tau : M \to \mathbb{R}$ is a convex function
- $M_c \overset{\text{def}}{=\!=} \{x \in M : \tau(x) < c\}$ be the sublevel set
- $\gamma : [a,b] \to M$ be a geodesic.

Then $\gamma(a), \gamma(b) \in M_c \Rightarrow \gamma([a,b]) \subset M_c$

(2) Furthermore, if M is complete, then M_c is totally convex.

Pf: (1) $\tau\circ\gamma(t) \le \max\{\tau\circ\gamma(a), \tau\circ\gamma(b)\} < c$
since $\tau\circ\gamma$ convex.

(2) Easily follows from (1).

Cor (of Thm3) Geodesic balls of a simply-connected complete Riem. mfd M with nonpositive sectional curvature are totally convex.

In particular, $\forall x \in M$, $\{x\}$ is totally convex. Therefore, there is no nontrivial geodesic $\gamma:[a,b]\to M$ s.t.

$$\gamma(a)=\gamma(b)=x.$$

Thm6 (J.H.C. Whitehead) Let M = Riem. mfd. Then $\forall x\in M$, $\exists$ a convex nbd. of x.

Pf: $\forall x\in M$, Lemma 4 (& properties of $\exp_x$)

$\Rightarrow \exists \varepsilon>0$ s.t.

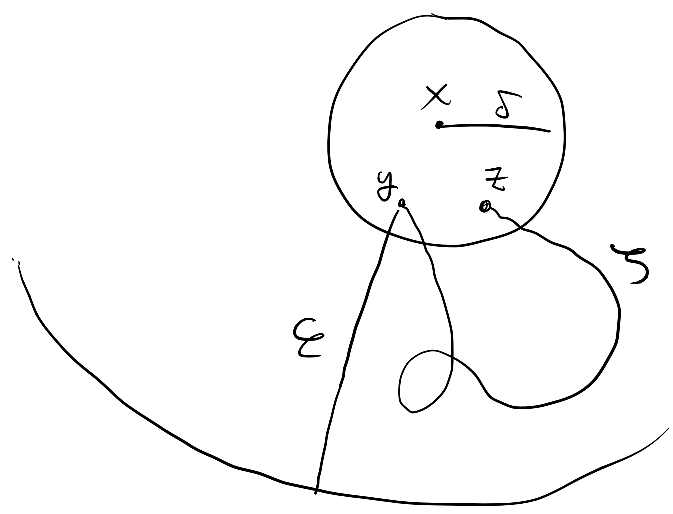
$$\begin{cases} \bullet\ \exp_x: \underset{\subset T_xM}{B(\varepsilon)} \longrightarrow \underset{\subset M}{B_\varepsilon(x)} \text{ is a diffeomorphism} \\ \bullet\ B_\varepsilon(x) = \exp_x(B(\varepsilon)) \text{ has compact closure in } M \\ \qquad\qquad \text{(note: } M \text{ may not be complete)} \end{cases}$$

1. ρ^2 is C^∞ & $D^2\rho^2 > 0$ on $B_\varepsilon(x)$,

where ρ = distance to x.

In fact, by choosing a smaller $\varepsilon > 0$, we can also assume that $\forall\, y \in B_\varepsilon(x)$, $\exp_y \big|_{B(\varepsilon)}$ is a diffeomorphism.

Let $\delta = \varepsilon/3 > 0$ and consider the geodesic ball $B_\delta(x)$. We claim that $B_\delta(x)$ is convex.



$\forall$ fixed $y \in B_\delta(x)$, we observe that $B_\delta(x) \subset B_\varepsilon(y)$.

In fact, $\forall\, z \in B_\delta(x)$,

$$d(z,y) \le d(z,x) + d(x,y) \le \delta + \delta = 2\delta = \frac{2\varepsilon}{3} < \varepsilon$$

$\therefore\ B_\delta(x) \subset B_\varepsilon(y)$

Therefore, $\forall z \in B_\delta(x)$, $\exists$ shortest geodesic ζ joining z to y with $\zeta \in B_\varepsilon(y)$ and $L(\zeta) < \varepsilon$.

However, we must have $\zeta \subset B_\varepsilon(x)$.

Otherwise, $y, z \in B_\delta(x) \Rightarrow$

$$L(\zeta) > 2(\varepsilon - \delta) = 2(\varepsilon - \tfrac{\varepsilon}{3}) = \tfrac{4}{3}\varepsilon > \varepsilon$$

which is a contradiction.

Since $D^2\rho^2 > 0$ on $B_\varepsilon(x)$, statement (1) of lemma 5

on $B_\delta(x)$ $(\subset B_\varepsilon(x)) \Rightarrow$

$B_\delta(x) =$ sublevel set of ρ^2

$\Rightarrow$ $\zeta \subset B_\delta(x)$ since ζ is the shortest geodesic joining z to y.

Since $y \in B_\delta(x)$ is arbitrary, we've shown that $\forall y, z \in B_\delta(x)$, $\exists$ shortest geodesic $\zeta \subset B_\delta(x)$ joining y & z. $\therefore$ $B_\delta(x)$ is convex. $\#$

Application 2 : Synge Thm

Fact : • A C^∞ mfd M of n-dim. is said to be orientable $\Leftrightarrow$ $\exists$ a nowhere zero C^∞ n-form ω on M

(i.e. $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ in local coordinates.
Alternating $(0,n)$-tensor : $\omega(X_1, \cdots X_i, \cdots X_j, \cdots, X_n)$
$= -\omega(X_1, \cdots X_j, \cdots, X_i, \cdots X_n)$)

• If such an ω is chosen, then it is called the orientation of M ($\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 = f\omega_2$ for some function $f > 0$)

• Let ω be a nowhere zero n-form on such an M, then bases of T_xM can be divided into 2-classes:

- positive oriented : $\omega(e_1, \cdots, e_n) > 0$
- negative oriented : $\omega(e_1, \cdots, e_n) < 0$.

(wrt ω)

Lemma 7: Let $\gamma : [a,b] \to M$ be a closed curve in an orientable Riem. mfd M such that $x = \gamma(a) = \gamma(b)$.

Then the parallel transport along γ

$$P^\gamma : T_xM \to T_xM \quad \text{has} \quad \det P^\gamma = +1.$$

(Pf. Ex!)

Lemma 8: Let M = non-simply-connected compact Riem mfd. ($\pi_1(M) \neq 1$)

Then $\exists$ closed curve $\gamma : [0,b] \to M$ (for some $b > 0$) such that $L(\gamma) \leq L(\alpha)$ for any piecewise C^∞ closed curve α which is non-homotopic to zero (i.e. $\alpha \not\simeq 1$)

(Pf: Omitted)

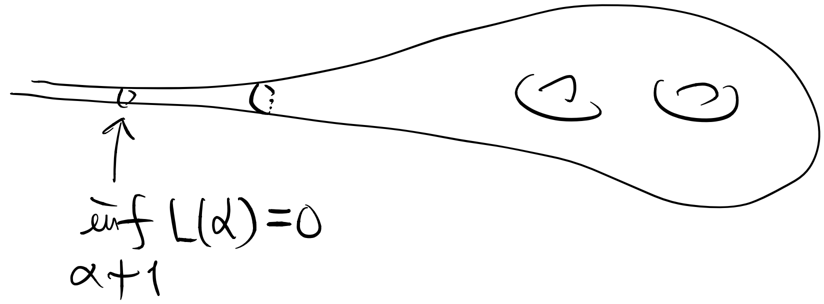
Notes: • $\pi_1(M) \neq 1$ is necessary: otherwise any closed curve can be shrinked to a point.

$\Rightarrow \inf_\gamma L(\gamma) = 0$

$\Rightarrow$ no curve minimizes the length functional.

• compactness is also necessary:

eg: surface with a cusp:



Thm 9 (J. L. Synge) If M is a compact orientable even dim'l Riem. mfd with positive sectional curvature, then M is simply-connected.

Pf: Suppose not, then $\pi_1(M) \neq 1$.

By Lemma 8, $\exists$ a closed curve $\gamma : [0,b] \to M$ s.t. $L(\gamma) \leq L(\alpha)$, $\forall \alpha \neq 1$.

Then γ has to a geodesic and hence

$$\gamma'(0) = \gamma'(b).$$

We may also assume $|\gamma'(t)| = 1$.

Let $x = \gamma(0) = \gamma(b)$. Then parallel transport along γ: $P^\gamma : T_xM \to T_xM$ has $\det P^\gamma = +1$ (Lemma 7).

Note that eigenvalues of P^γ are of the form ± 1, $e^{i\theta}$ $(\neq \pm 1)$, and if $e^{i\theta}$ is an eigenvalue, then $e^{-i\theta}$ is also an eigenvalue.

Since $\det P^\gamma = +1$, the $\dim\{-1 \text{ eigenspace}\}$ is even

Hence $\dim M = \text{even} \Rightarrow \dim\{+1 \text{ eigenspace}\}$ is also even.

Note that γ is a closed geodesic, $\gamma'(0) = \gamma'(b)$

& $\quad P^\gamma(\gamma'(0)) = \gamma'(b) = \gamma'(0)$

$\Rightarrow \quad \dim\{+1 \text{ eigenspace}\} > 0$, hence ≥ 2.

Therefore, $\exists\, e \in T_xM$ s.t. $P^\gamma(e) = e$
and $\langle e, \gamma'(0) \rangle = 0$.

Now, let V be the parallel vector field along γ such that $V(0) = e$.

Then $V(b) = P^\gamma(V(0)) = P^\gamma(e) = e$

$\Rightarrow$ V is well-defined vector field on the closed curve γ.

$\Rightarrow$ $\exists$ a 1-parameter family of closed geodesics

$\{\gamma_u\}$ s.t. $\gamma_0 = \gamma$ & V = transversal vector field of $\{\gamma_u\}$ $\left(\gamma_u(t) = \exp_{\gamma(t)}(u V(t)),\ |u| \ll 1\right)$.

Then 2^{nd} variation formula $\Rightarrow$

$$\frac{d^2 L}{du^2}(0) = \int_0^b \left[|D_{\gamma'} V^\perp|^2 - \langle R_{\gamma' V^\perp} \gamma', V^\perp \rangle \right] dt$$

(since γ_u closed $\forall u \Rightarrow$ bdy term $=0$)

Since $\langle V(0), \gamma'(0) \rangle = \langle e, \gamma'(0) \rangle = 0$, we have

$$V^\perp = V, \quad \forall t \in [0,b]$$

$$\Rightarrow \quad D_{\gamma'} V^\perp = D_{\gamma'} V = 0 \quad \text{(since } V \text{ parallel)}$$

$$\therefore \quad \frac{d^2 L}{du^2}(0) = -\int_0^b \langle R_{\gamma' V^\perp} \gamma', V^\perp \rangle dt < 0$$

(since sectional curvature > 0)

Contradicting that γ is length minimizing.

$\therefore$ $\pi_1(M) = 1$.

$\divideontimes$