§5.2 Geodesic & Curvatures

Let $\mathbb{H}^n = (\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i)$

Facts: $\mathbb{R}^2 \hookrightarrow \mathbb{R}^n$, $S^2 \hookrightarrow S^n$, $\mathbb{H}^2 \hookrightarrow \mathbb{H}^n$ are totally geodesic submanifolds, the studies of geodesics on $\mathbb{R}^n$, S^n & $\mathbb{H}^n$ can be reduced to $\mathbb{R}^2$, S^2, & $\mathbb{H}^2$.

(Since $\forall x, y \in \mathbb{R}^n$, S^n or $\mathbb{H}^n$, $\exists$ isometry of $\mathbb{R}^n$, S^n or $\mathbb{H}^n$ respectively, taking x to y. (Ex))

Let $M = \mathbb{R}^2, S^2$, or $\mathbb{H}^2$, and let $O \in M$ be a fixed point.

Let $C(r) = \{ x \in M : d(O, x) = r \}$ be the geodesic circle of radius r.

If $r > 0$ small enough, then

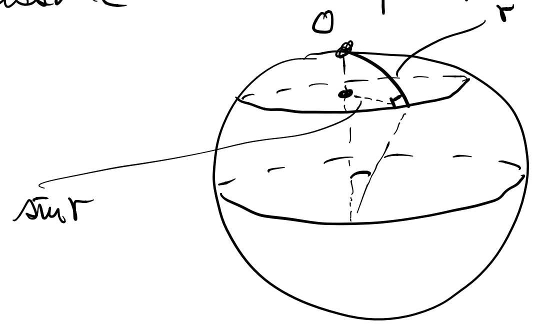
$$C(r) = \exp_O(\text{circle of radius } r \text{ in } T_O M)$$

Denote
$$\text{length } C(r) = \begin{cases} C_0(r) & , \text{if } M = \mathbb{R}^2 \\ C_+(r) & , \text{if } M = S^2 \\ C_-(r) & , \text{if } M = \mathbb{H}^2 \end{cases}$$

If $M=\mathbb{R}^2$, it is clear that

$$\boxed{C_0(r) = 2\pi r}$$

If $M = S^2$, we may assume O = north pole.



Then the geodesic circle

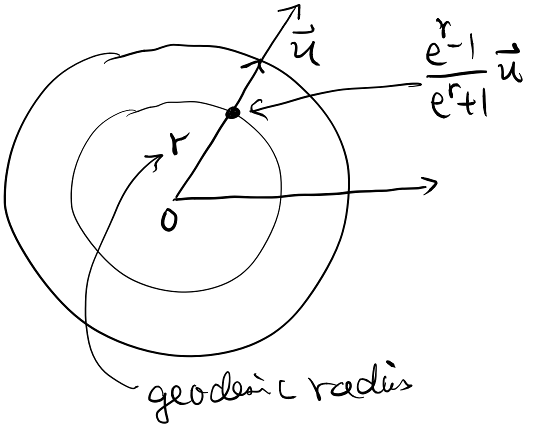
$C(r)$ = a circle

of radius $\sin r$ in $\mathbb{R}^3$

$$\Rightarrow \boxed{C_+(r) = 2\pi \sin r}$$

If $M=\mathbb{H}^2$, then by the proof of Lemma 6, a normalized geodesic from O is given by

$$\gamma(s) = \frac{e^s-1}{e^s+1}\vec{u}, \text{ where } \vec{u} = \text{unit vector in } \mathbb{R}^2$$



where s = arc-length

$\left(|\gamma'(s)|_{\mathbb{H}^2} = 1 \right)$

$$\Rightarrow d_{\mathbb{H}^2}(0,\gamma(r)) = \int_0^r |\gamma'(s)|_{\mathbb{H}^2} ds$$
$$= r$$

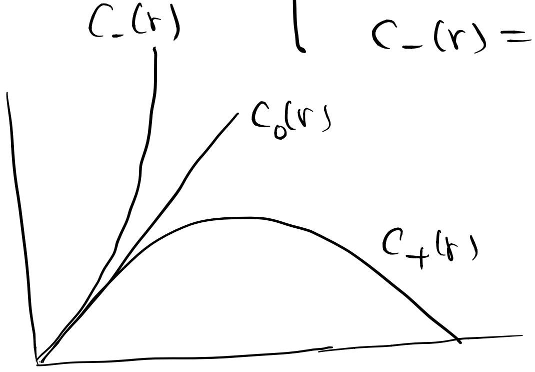
$\Rightarrow C(r)$ = Euclidean circle of radius $\frac{e^r-1}{e^r+1}$ $\left(= \tanh\frac{r}{2}\right)$

$$\Rightarrow \quad C_-(r) = \int_0^{2\pi} \frac{2}{1-\rho^2}\,\rho\,d\theta \qquad \text{where } \rho = \tanh\frac{r}{2}$$

$$= 2\pi \cdot \frac{2\rho}{1-\rho^2}$$

$$\Rightarrow \quad \boxed{C_-(r) = 2\pi \sinh r}$$

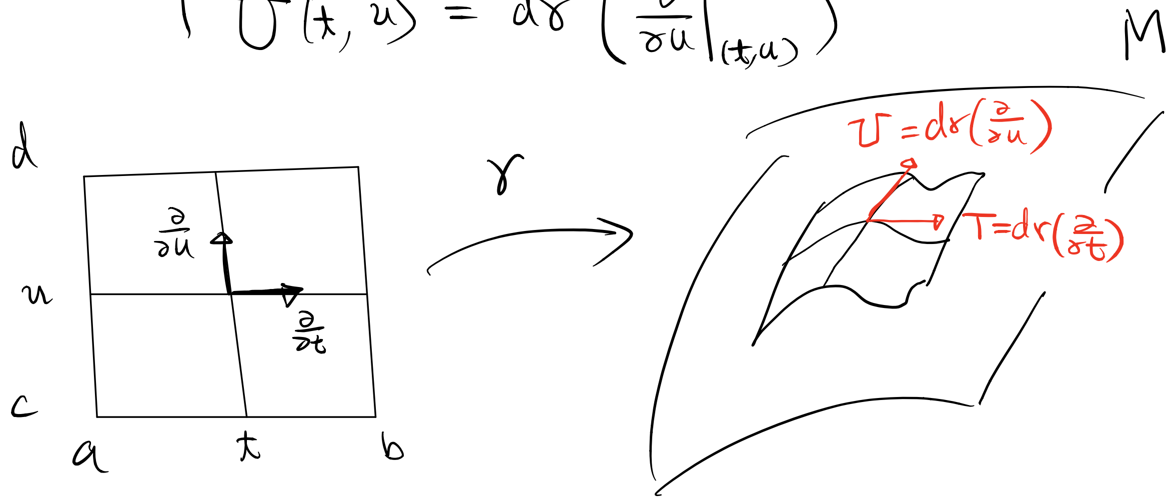
In summary, we have $\begin{cases} C_0(r) = 2\pi r \\ C_+(r) = 2\pi \sin r \\ C_-(r) = 2\pi \sinh r \end{cases}$.



To generalize the above to arbitrary <u>complete</u> Riem. manifold, we need to study <u>variations</u> of geodesics.

Let $\gamma : [a,b] \times [c,d] \to M$ be a C^∞ map from the rectangle $[a,b] \times [c,d]$ to a complete Riem. manifold M (of dim ≥ 2). Denote a point in $[a,b] \times [c,d]$ by (t, u). Then we can define 2 tangent vecto fields along γ by

$$\left\{\begin{array}{l} T(t,u) = d\gamma\left(\frac{\partial}{\partial t}\Big|_{(t,u)}\right) \\ U(t,u) = d\gamma\left(\frac{\partial}{\partial u}\Big|_{(t,u)}\right) \end{array}\right.$$



$\forall$ fixed $u \in [c,d]$, a curve

$$\begin{array}{ccccc} \gamma_u = & [a,b] & \longrightarrow & M & \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & t & \longmapsto & \gamma(t,u) & \end{array}$$

is defined.

Suppose $0 \in [c,d]$. Then γ_0 is called the base curve of γ. If γ_u are geodesics $\forall\, u \in [c,d]$, we call γ a one-parameter family of geodesics.

In this case, the vector field $T = \gamma_u'$ and hence

$$D_T T = 0 .$$

We also have $[T,U] = d\gamma\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right) = 0$

Hence $\begin{cases} [T,U] = 0 \\ D_T T = 0 \end{cases}$ along γ.

Then
$$D_T D_T U = D_T(D_U T)$$
$$= D_T(D_U T) - D_U(\cancelto{0}{D_T T}) - D_{\cancelto{0}{[T,U]}} T$$
$$= -R_{TU}T$$

Therefore, along the base geodesic γ_0, we have

$$\boxed{D_{\gamma_0'} D_{\gamma_0'} U + R_{\gamma_0' U}\gamma_0' = 0} \quad \text{(Jac)}$$

or simply $\boxed{U'' + R_{\gamma_0' U}\gamma_0' = 0}$

by writting $U'' = D_{\gamma_0'} D_{\gamma_0'} U$ (similarly $U' = D_{\gamma_0'} U$)

Def :
- Equation (Jac) is called the *Jacobi equation* along γ_0.
- Solutions of (Jac) are called *Jacobi fields* along γ_0.

Note : The vecta field U constructed above is called a <u>transversal vecta field</u> (a <u>variational vecta field</u>) of $\{\gamma_u\}$.

<u>Lemma 7</u>: A transversal vecta field of a 1-parameter family of geodesics is a Jacobi field.

<u>eg</u> : If M = 2 dim'l complete Riem. manifold.

Denote $C'(r) = \{ x \in M : d(x, o) = r \}$

$c(r) =$ length $C'(r)$

where $O \in M$ is fixed.

Let (ρ, θ) = polar coordinates on $T_o M$.

Let $\delta > 0$ small s.t. $\exp_o$ is a diffeomaphism on

$$B(\delta) = \{ v \in T_o M : \rho(v) < \delta \}$$

We can parametrize a circle of radius r in $B(\delta)$ by

$$\begin{array}{ccc} \tilde{r} : [0, 2\pi] & \longrightarrow & B(\delta) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \theta & \longmapsto & (r, \theta) \end{array}$$

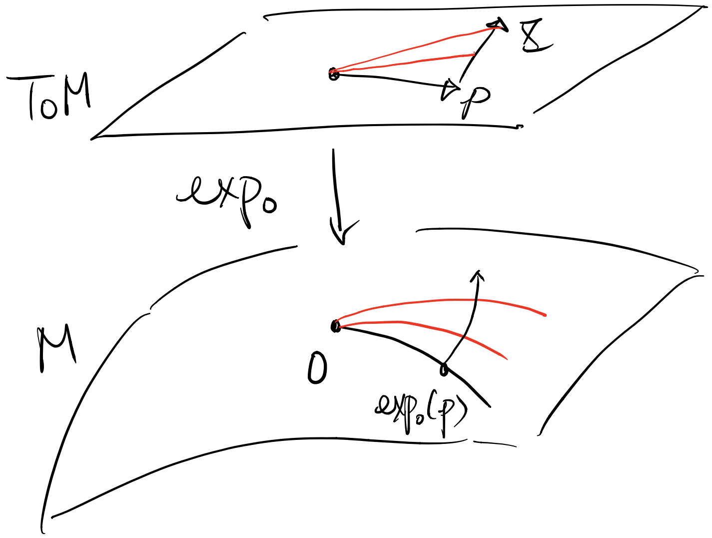
Then $C'(r) = \exp_o(\tilde{r})$ and

$$c(r) = \int_0^{2\pi} \left| (d\exp_o)_{(r,\theta)} \left(\frac{\partial}{\partial\theta}\right) \right| d\theta$$

Note that $(d\exp_o)_{(r,\theta)}(\frac{\partial}{\partial\theta})$ is a transversal vector field of the family of radial geodesis (with <u>specific initial values</u>).

<u>General setting</u>

Let
- M = complete Riem. manifold of dim $n \geq 2$
- $o \in M$ fixed point
- $p \in T_oM$
- $X \in T_p(T_oM) \cong T_oM$



Define $\Gamma = [0,r]\times[0,1] \to M$, where $r = |p|$ by

$$\Gamma(t,u) = \exp_0\left[\frac{t}{r}(p+uX)\right]$$

Then $\forall\, u\in[0,1]$, $\Gamma_u(t) = \Gamma(t,u)$ is a geodesic with initial tangent vector $\frac{1}{r}(p+uX)$ (not of length 1 unless $u=0$)

$\Rightarrow \Gamma(t,u)$ is a 1-parameter family of geodesics.

Let $V(t)$ = transversal vector field along Γ_0, and $\delta > 0$ be a number s.t. $\exp_0$ is a diffeo on

$$B(\delta) = \{v\in T_0M : |v|<\delta\} \quad (|v| = \rho(v) \text{ in polar coordinates})$$

Set $B_\delta = \{x\in M : d(0,x)<\delta\}$. Then

$$B_\delta = \exp_0(B(\delta)).$$

Let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of T_0M and $\{\alpha^1,\cdots,\alpha^n\}$ be the dual basis of $\{e_1,\cdots,e_n\}$.

Then $\{\alpha^1,\cdots,\alpha^n\}$ are coordinate functions on T_0M.

Define a coordinate system on B_δ by

$$x^i = \alpha^i\circ\exp_0^{-1} : B_\delta \to \mathbb{R} \quad (i=1,\cdots,n).$$

Then

<u>Claim</u>

$$\begin{cases} \left\langle \frac{\partial}{\partial x^i}\Big|_0 , \frac{\partial}{\partial x^j}\Big|_0 \right\rangle = \delta_{ij} , & \forall i,j \\[2ex] D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0 , & \forall i,j \end{cases}$$

(Note: coordinate systems satisfying these coordinates are called normal coordinate systems.)

<u>Pf</u>: The 1st eqt follows from $\left(d\exp_{\underset{\substack{\nearrow \\ 0\in M}}{0}}\right)_{\underset{\substack{\nwarrow \\ 0\in T_0M}}{0}} = Id$.

To see the 2nd, we define a bilinear form

$$\beta : T_0M \times T_0M \to \mathbb{R}^n$$

by $$\beta(e_i, e_j) = D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} .$$

Then $\forall\, v = \sum v^i e_i \in T_0M$,

$$\beta(v,v) = \sum_{i,j} v^i v^j \beta(e_i, e_j) = \sum_{i,j} v^i v^j D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j}$$

$$= D_{\left(\sum_i v^i \frac{\partial}{\partial x^i}\right)\big|_0} \left(\sum_j v^j \frac{\partial}{\partial x^j} \right) .$$

Note that $\sum v^j \frac{\partial}{\partial x^j}\Big|_0$ is the initial tangent vector of the geodesic $\exp_0(t\sum_i v^i e_i)$. Hence $\beta(v,v)=0$ by the geodesic eqt.

$\Rightarrow$ $\beta \equiv 0$ on T_0M

i.e. $D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0 \quad \forall i,j$

※

Now assume $p=\sum p^i e_i$ & $X = \sum X^i e_i$ (under $T_p(T_0M)\doteq T_0M$)

For $\varepsilon>0$ small, εp & $\varepsilon X \in B(\delta)$.

Then in the above coordinate system $\{x^1,\cdots,x^n\}$, the coordinate vector of $\Gamma(t,u)=\exp_0\left(\frac{t}{r}(p+uX)\right)$ is

$$\frac{t}{r}(\vec{p}+u\vec{X}), \text{ where } \vec{p}=\begin{pmatrix}p^1\\ \vdots\\ p^n\end{pmatrix} \text{ and } \vec{X}=\begin{pmatrix}X^1\\ \vdots\\ X^n\end{pmatrix},$$

for $(t,u)\in[0,\varepsilon r]\times[0,\varepsilon]$.

And the base geodesic is $\Gamma_0(t)=\Gamma(t,0) \overset{\text{in coordinates}}{=\!=} \frac{t}{r}\vec{p}$.

$\Rightarrow$ $U(t)=\frac{\partial}{\partial u}\Gamma(t,u) = \frac{t}{r}\vec{X}$ (in coordinates)

i.e. $U(t)=\frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}$

Therefore, $V(0)=0$ and

$$V'(0) = D_{\Gamma_0'(0)} V = \frac{d}{dt}\Big|_{t=0}\left(\frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}\right)$$

$$= \frac{1}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_0 + 0 .$$

In conclusion, the transversal vector field $V(t)$ of $\Gamma(t,u) = \exp_0\left[\frac{t}{r}(p+uX)\right]$ satisfies

$$\begin{cases} V(0)=0 \\ V'(0) = \frac{1}{r}\vec{X} \text{ (in coordinate), where } r=|p|. \end{cases}$$

[Note: $V(t) = \frac{t}{r}(d\exp_0)_{(\frac{t}{r}p)}(X)$ (check!)]

Applying the above to $M = \mathbb{R}^2, S^2$ or $\mathbb{H}^2$ with $p=(r,\theta)$, $X = \frac{\partial}{\partial\theta}\Big|_{(r,\theta)}$. Then $V(r) = (d\exp_0)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ (at $t=r$) is a Jacobi field satisfying

$$\begin{cases} V(0)=0 \\ |V'(0)| = \frac{1}{r}\left|\frac{\partial}{\partial\theta}\right| = 1 \end{cases} \quad ((r,\theta) = \text{polar coordinates}).$$

Let $W(t)$ = <u>unit</u> parallel vector field along Γ_0 s.t.

$$\langle W(t), \Gamma_0'(t) \rangle = 0$$

On the other hand, Gauss lemma

$\Rightarrow$ $\quad V(t) = (d\exp_0)_{(t,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ is normal to $\Gamma_0'(t)$.

In our case of $\dim M = 2$,

$$V(t) = (d\exp_0)_{(t,\theta)}\left(\frac{\partial}{\partial\theta}\right) = f(t)W(t)$$

for some function $f \in C^\infty[0,r]$.

Then $\quad V'(t) = D_{\Gamma_0'(t)}V(t) = f'(t)W(t)$ and

$$V''(t) = D_{\Gamma_0'(t)}D_{\Gamma_0'(t)}V(t) = f''(t)W(t)$$

(since W is parallel).

$$(\text{Jac}) \Rightarrow \quad f''(t)W(t) + R_{\Gamma_0', fW}\Gamma_0' = 0$$

$$\Rightarrow \quad f''(t) + f\langle R_{\Gamma_0' W}\Gamma_0', W\rangle = 0$$

i.e. $\quad f'' + Kf = 0$, where K = Gauss curvature at $\Gamma_0(t)$

(in general $K=$ sectional curvature $(\text{span}\langle\Gamma_0', W\rangle)$
since $|\Gamma_0'(t)|=|W(t)|=1$ & $\langle\Gamma_0', W\rangle=0$)

We may also assume $\langle W, \frac{\partial}{\partial\theta}\rangle>0$, we have

$$\begin{cases} f''+Kf=0 \\ f(0)=0 \\ f'(0)=1 \end{cases}$$

$\therefore$ The signature of K has implication on

$$c(r)=\int_0^{2\pi}\left|(d\exp_o)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)\right|d\theta=\int_0^{2\pi}f\,d\theta.$$

In particular, if $K=0,\pm1$, we have

$$f(r)=\begin{cases} r & , \quad K\equiv0 \\ \sin r & , \quad K\equiv+1 \\ \sinh r & , \quad K\equiv-1. \end{cases}$$

Prop : Let $K\geq+1$, then $c(r)\leq2\pi\sin r$, for small r.

Pf : Consider a comparison function $h(t)=\sin t$.

Then $\begin{cases} h''+h=0 \\ h(0)=0 \\ h'(0)=1 \end{cases}$

$$\Rightarrow\ (hf'-fh')'=hf''-fh''=-Kfh+fh$$

$$= -(K-1) f h$$

Since $f(0)=h(0)=0$, $f'(0)=h'(0)=1$, we have

$$f \geq 0,\ h \geq 0 \quad \text{for small } t>0.$$

$$\Rightarrow \quad (hf'-fh')' \leq 0 \quad \text{for small } t>0$$

$$\Rightarrow \quad hf'-fh' \leq h(0)f'(0)-f(0)h'(0)=0 \quad \text{for small } t>0.$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' = \frac{hf'-fh'}{h^2} \leq 0 \quad \text{for small } t>0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \leq \lim_{\varepsilon>0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1 \quad \text{for small } t>0$$

$$\Rightarrow \quad f(t) \leq h(t) = \sin t \quad \text{for small } t>0$$

Hence $\displaystyle c(r) = \int_0^{2\pi} f(r,\theta)\,d\theta \leq 2\pi \sin r$ for small $r>0$.

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<u>Prop</u>: If $K \leq -1$, we have $c(r) \geq 2\pi \sinh r$

(for small r at this moment)

<u>Pf</u>: Consider $h(t) = \sinh t$

Then $\begin{cases} h''-h=0 \\ h(0)=0 \\ h'(0)=1 \end{cases}$

$$\Rightarrow \quad (hf'-fh')' = hf''-fh'' = -Kfh - fh$$

$$= -(K+1) fh$$
$$\geq 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad hf' - fh' \geq h(0)f'(0) - f(0)h'(0) = 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' \geq 0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \geq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

$$\Rightarrow \quad f(t) \geq \sinh(t) \quad \text{for small } t > 0 .$$

$$\Rightarrow \quad C(r) \geq 2\pi \sinh(r), \quad \text{for small } r > 0 .$$

#

Ch6 Jacobi Field, Cartan-Hadamard Thm

§6.1 Jacobi Field

Let γ = normalized geodesic (ie. $|\gamma'|=1$)

Recall that the Jacobi equation (for vector field along γ) is

$$U'' + R_{\gamma' U}\gamma' = 0 \qquad (\text{Jac})$$

where $U'' = D_{\gamma'}D_{\gamma'}U \quad (U' = D_{\gamma'}U)$

Let $\{e_1(t), \cdots, e_n(t)\}$ be parallel vector fields along γ such that $\forall t$

$$\begin{cases} e_1(t) = \gamma'(t) \\ \{e_i(t)\}_{i=1}^{n} \text{ is an orthonormal basis of } T_{\gamma(t)}M. \end{cases}$$

Then $\forall$ vector field U along γ, we write

$$U(t) = \sum_{i=1}^{n} f^i(t)\, e_i(t), \quad \text{for some functions } f^i(t).$$

Similarly, the curvature can be written as

$$R_{e_i(t)\, e_j(t)}\, e_k(t) = \sum_{l=1}^{n} R_{ijk}^{l}\, e_l(t),$$

where $R_{ijk}^{l}(t) = \langle R_{e_i(t) e_j(t)}\, e_k(t), e_l(t)\rangle$.

Then the eqt. (Jac) $\Rightarrow$

$$0 = V'' + R_{\gamma' V}\gamma'$$

$$= \left(\sum f^i e_i\right)'' + R_{e_1 (\sum f^l e_l)} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l R_{e_1 e_l} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l \left(\sum_i R^i_{1l1} e_i\right)$$

$$= \sum_i \left[(f^i)'' + \sum_l R^i_{1l1} f^l\right] e_i$$

$\therefore$ (Jac) $\Leftrightarrow$ $\boxed{(f^i)'' + \sum_l R^i_{1l1} f^l = 0, \ \forall i = 1, \cdots, n}$

which is a 2^{nd} order linear ODE system.

Then ODE theory $\Rightarrow$

Lemma 1

(1) Let γ be a geodesic. Then given any $v, w \in T_{\gamma(0)}M$, $\exists$ a unique Jacobi field $V(t)$ along γ s.t.

$$V(0) = v, \quad V'(0) = w.$$

(2) Unless $V \equiv 0$, the zero set of $V(t)$ along γ is

discrete.

(Pf: by standard ODE theory)

<u>Lemma 2</u> Let U be a vector field along a normalized geodesic γ. Then U is a Jacobi field along γ $\Leftrightarrow$ U is the transversal vector field of a one-parameter family of geodesics.