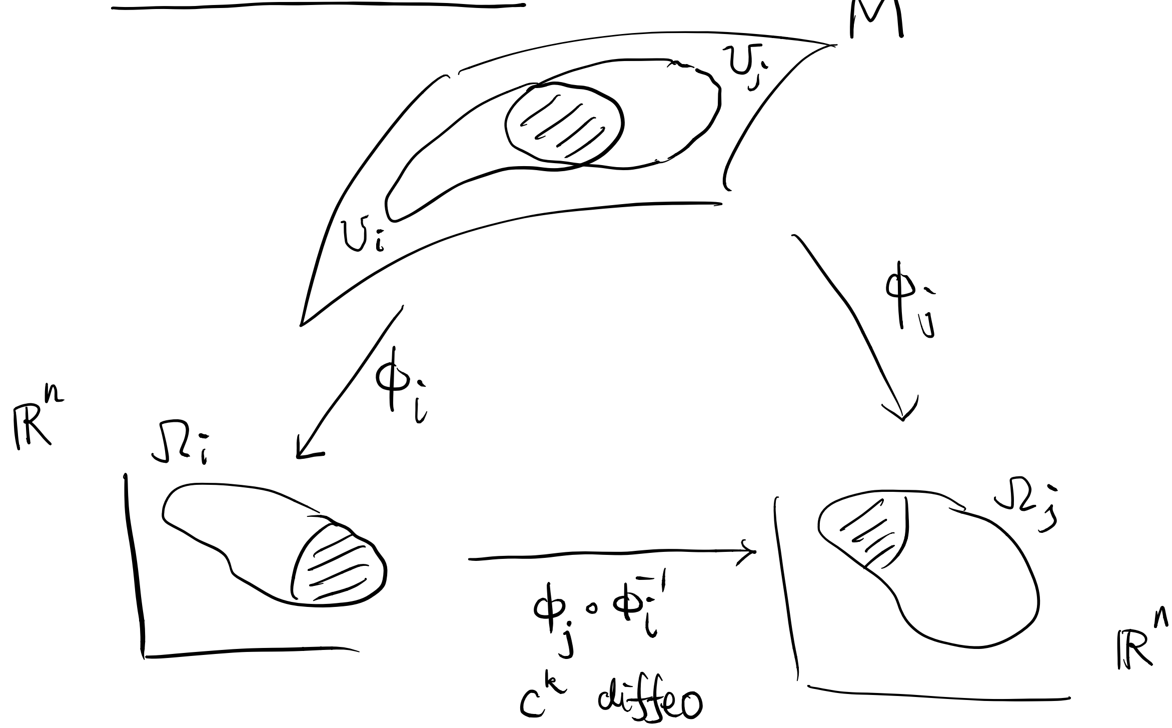
Ch1 Differential Manifolds

1.1 Abstract Manifolds



Def: A C^k atlas on a Hausdroff topological space M is given by

(i) an open covering U_i, $i \in \Lambda$, of M;

(ii) a family of homeomaphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad (\Omega_i \text{ is open})$$

such that $\forall\, i, j \in \Lambda$,

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^k-diffeomaphism.

Remark : • $\phi_j \circ \phi_i^{-1}$, $i,j \in \Lambda$ (with $U_i \cap U_j \neq \emptyset$) are called transition functions.

• (U_i, ϕ_i) is a called a (coordinate) chart.

• $\phi_i^{-1} : \Omega_i \to U_i \subset M$ is a local parametrization.

Def : Two C^k altas on M, say $(U_i, \phi_i)_{i\in\Lambda_1}$ and $(V_j, \psi_j)_{j\in\Lambda_2}$ are C^k-equivalent if their union is still a C^k altas, that is, if $\forall\, i \in \Lambda_1, j \in \Lambda_2$ (s.t. $U_i \cap V_j \neq \emptyset$)

$$\phi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$$

are C^k diffeomorphisms.

Def : A differentiable structure of class C^k on M is an equivalence class of C^k altas.

Remark : If M is connected, then integer n in the definition doesn't depend on the chart and is defined as the dimension of M.

Def : A C^k differentiable manifold of dimension n is a pair $(M, \mathcal{A})$, where M is a Hausdroff top. space and $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in\Lambda}$ is a C^k altas on M with $\phi_i(U_i) \subset \mathbb{R}^n$.

Remark: • In this course, we consider only C^∞ differentiable manifold which is connected and with a further condition such that "partitions of unity" is always possible.

• All compact manifolds satisfy the further condition.

• We'll refer such a manifold as a smooth manifold (or even just manifold).

eg: $M = T^n$, the n-torus $\left(T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} \right)$

Pf: Let $f : \mathbb{R}^n \to T^n \in \mathbb{C}^n$ (f is onto)

$$(x_1, \cdots, x_n) \mapsto (e^{ix_1}, \cdots, e^{ix_n})$$

$\forall\ p \in T^n$, $\exists\ x^p = (x_1^p, \cdots, x_n^p) \in \mathbb{R}^n$ such that

$$p \in f(x^p) \quad \left(\text{We may choose } x_i^p \in [0, 2\pi),\ i = 1, \cdots, n\right)$$

Consider $\Omega_p = (x_1^p - \pi, x_1^p + \pi) \times \cdots \times (x_n^p - \pi, x_n^p + \pi)$ open in $\mathbb{R}^n$ containing x^p.

Let $U_p = f(\Omega_p) \subset T^n$ (U_p open & containing p)

$$\phi_p = \left(f|_{\Omega_p} \right)^{-1} : U_p \to \Omega_p \subset \mathbb{R}^n \text{ homeo.}$$

Then $\{(U_p, \phi_p)\}_{p \in T^n}$ is an C^∞ altas on T^n:

In fact, if $p, q \in T^n$ s.t. $U_p \cap U_q \neq \emptyset$,

then $\phi_q \circ \phi_p^{-1}(x_1, \cdots, x_n) \quad \left((x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q) \subset \Omega_p \right)$

$$= \phi_q(f(x_1, \cdots, x_n))$$

$$= \phi_q(e^{ix_1}, \cdots, e^{ix_n}) \qquad \left((e^{ix_1}, \cdots, e^{ix_n}) \in U_p \cap U_q \subset U_q \right)$$

$$= (f|_{\Omega_q})^{-1}(e^{ix_1}, \cdots, e^{ix_n})$$

$$= (x_1 + 2k_1\pi, \cdots, x_n + 2k_n\pi) \text{ for some } k_1, \cdots, k_n$$

such that $x_i + 2k_i\pi \in (x_i^q - \pi, x_i^q + \pi)$

note that k_i are indep of $(x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q)$

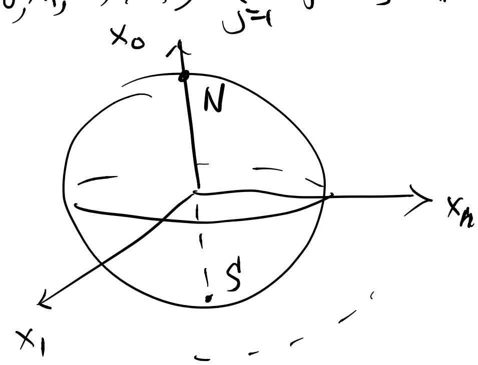
Hence $\phi_q \circ \phi_p^{-1}$ is just a <u>translation</u> in $\mathbb{R}^n$.

$\Rightarrow \phi_q \circ \phi_p^{-1}$ is a C^∞ diffeo.

$\therefore \left(T^n, \{(U_p, \phi_p)\}_{p \in T^n}\right)$ is a smooth manifold.

eg $M = S^n$, the n-sphere $S^n = \{(x_0, x_1, \cdots, x_n) : \sum_{j=1}^{n} x_j^2 = 1\} \subset \mathbb{R}^{n+1}$

Let $\begin{cases} N = (1, 0, \cdots, 0) \in S^n \\ S = (-1, 0, \cdots, 0) \in S^n \end{cases}$



$\begin{cases} U_1 = S^n \setminus \{N\} \\ U_2 = S^n \setminus \{S\} \end{cases}$

$U_1 \cup U_2 = S^n$

Let

$$\begin{cases} \phi_1 : U_1 \longrightarrow \mathbb{R}^n \quad \text{(Stereographic projections)} \\ \quad\ \ \rotatebox{90}{$\in$} \qquad\qquad\quad \rotatebox{90}{$\in$} \\ (x_0, x_1, \cdots, x_n) \longmapsto \frac{1}{1-x_0}(x_1, \cdots, x_n) \\ \\ \phi_2 : U_2 \longrightarrow \mathbb{R}^n \\ \quad\ \ \rotatebox{90}{$\in$} \\ (x_0, x_1, \cdots, x_n) \longmapsto \frac{1}{1+x_0}(x_1, \cdots, x_n) \end{cases}$$

are homeomaphism

Note that if $\phi_1(x_0, x_1, \cdots, x_n) = (y_1, \cdots, y_n) \neq 0$

then $y = (y_1, \cdots, y_n) \in \phi_1(U_1 \cap U_2)$

(Ex:) $\boxed{\phi_2 \circ \phi_1^{-1}(y) = \frac{y}{|y|^2}}$ $(\forall\, y \in \mathbb{R}^n \setminus \{0\})$

which is a C^∞ diffeomaphism.

$\Rightarrow$ $\mathcal{A} = \{(U_1, \phi_1), (U_2, \phi_2)\}$ is an C^∞ altas on S^n, therefore $(S^n, \mathcal{A})$ is a smooth manifold.

eg: $\mathbb{RP}^n$ the <u>real projective space</u> (in some books $= P^n\mathbb{R}$)

- As topological space

  $\mathbb{RP}^n =$ quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalent relation: $(x, y \in \mathbb{R}^{n+1} \setminus \{0\})$

$$x \sim y \iff \exists \lambda \neq 0 \in \mathbb{R} \text{ s.t. } x = \lambda y$$

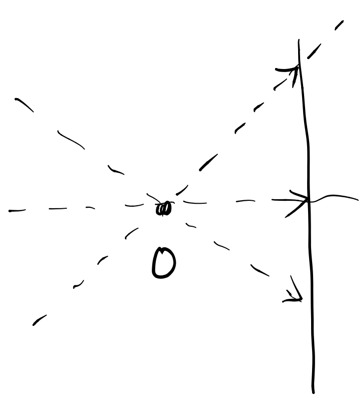
$$= S^n / \{\pm Id\} \quad \left(\text{hence } \mathbb{RP}^n = \text{Hausdroff, compact, connected.}\right)$$

- Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the canonical projection map, i.e. $\pi(x) =$ equi. class of x.

Define $V_i = \{ x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1},\ x_i \neq 0 \}$

$$\begin{aligned} \Phi_i : V_i &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \left(\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots \frac{x_n}{x_i} \right) \end{aligned}$$

this means the term is deleted.



$\{x_i = 1\} \cong \mathbb{R}^n$

Then $\forall\, x, y \in V_i$, we have

$$\boxed{\begin{aligned} &\Phi_i(x) = \Phi_i(y) \\ \iff &\pi(x) = \pi(y) \end{aligned}} \quad (*)$$

(ie. $x \sim y$)

This gives
$$\begin{tikzcd} \mathbb{R}^{n+1} \setminus \{0\} \supset V_i \arrow[r, "\Phi_i"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R}^n \\ U_i = \pi(V_i) \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

$(\phi_i \circ \pi = \Phi_i)$

where ϕ_i is defined by

$$\begin{array}{ccc} \phi_i : U_i = \pi(V_i) & \longrightarrow & \mathbb{R}^n \\ \Psi & & \Psi \\ \text{equi. class of } x & \longmapsto & \Phi_i(x) \end{array}$$

(ϕ_i is well-defined because of (*))

Further $\phi_i : U_i \to \mathbb{R}^n$ is homeomorphism (check!)

with inverse

$$\phi_i^{-1}(y_0, \cdots, y_{n-1}) = \pi(y_0, \cdots y_{i-1}, 1, y_i, \cdots, y_{n-1})$$

Therefore, if $y_j \neq 0$, say for $j < i$, we have

$$\begin{aligned}(\phi_j \circ \phi_i^{-1})(y_0, \cdots, y_{n-1}) &= \phi_j(\pi(y_0, \cdots y_{i-1}, 1, y_i, \cdots, y_{n-1})) \\ &= \Phi_j(y_0, \cdots y_{i-1}, 1, y_i, \cdots, y_{n-1}) \\ &= \left(\frac{y_0}{y_j} \cdots \widehat{\frac{y_j}{y_j}} \cdots, \frac{1}{y_j}, \frac{y_i}{y_j} \cdots \frac{y_{n-1}}{y_j}\right)\end{aligned}$$

$\therefore$ $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^∞ diffeo.

(similarly for $i < j$). Hence $\mathbb{RP}^n$ with the altas

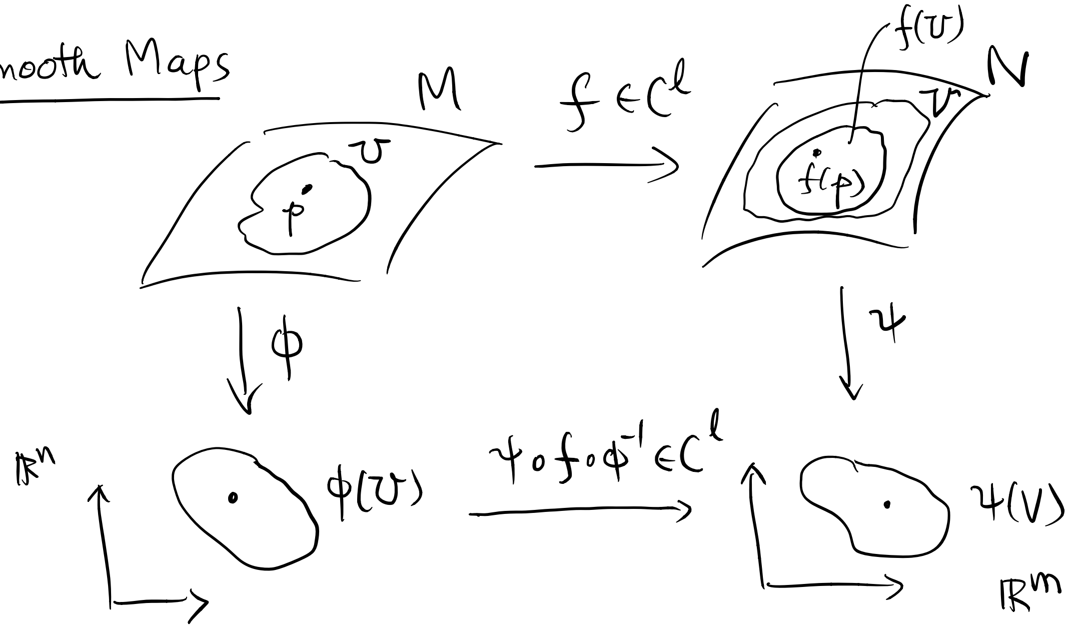
$\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth manifold.

Note: $\mathbb{RP}^n$ is non-orientable for n even according to the following definition: (proof omitted)

Def: A smooth manifold M is said to be orientable if $\exists$ an altas on M s.t. the Jacobian determinant

$$J(\phi_j \circ \phi_i^{-1}) > 0, \quad \forall i, j.$$

1.2 Smooth Maps



$\psi \circ f \circ \phi^{-1}$ is m functions of n variables as in Calculus

Def: Let M & N be C^k manifolds. A continuous map $f: M \to N$ is C^l map (for $l \leq k$) if $\forall p \in M$, $\exists$ charts (U, ϕ) & (V, ψ) for M and N around p & $f(p)$ respectively with $f(U) \subset V$ such that

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \text{ is } C^l.$$

Note: This definition doesn't depend on the charts since transition functions are C^k ($k \geq l$) (Ex!)

Def: A C^k map $\gamma: (a,b) \to M$ from an open interval to a smooth manifold is called a <u>C^k curve</u> (on M).

Def: A C^k map $f: M \to \mathbb{R}$ (or $\mathbb{C}$) is called a <u>C^k function</u> on M.

Def: A smooth map $g: \mathbb{R}^n \to \mathbb{R}^k$ is a <u>submersion</u> (an <u>immersion</u>, a <u>local diffeomorphism</u>) at $x \in \mathbb{R}^n$ if

$D_x g: \mathbb{R}^n \to \mathbb{R}^k$ is <u>surjective</u> (<u>injective</u>, <u>bijective</u>).

Def: Let M & N be smooth manifolds. A smooth map $f: M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo.</u>) at $p \in M$, if $\exists$ charts (U, ϕ) for M around p, (V, ψ) for N around $f(p)$, with $f(U) \subset V$ s.t.

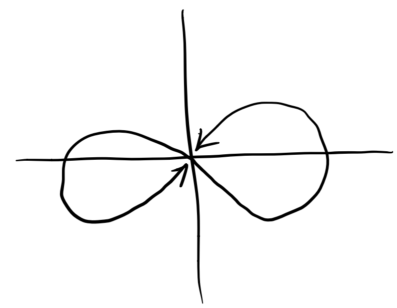
$\psi \circ f \circ \phi^{-1}$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo.</u>) at $\phi(p) \in \phi(U) \subset \mathbb{R}^n$.

Def: A map $f: M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeo.</u>) if it has the property at any point of M.

Def: A map $f: M \to N$ is a <u>diffeomorphism</u> if it is a <u>bijection</u> such that both <u>f and f^{-1} are smooth</u>.

Def: A map $f: M \to N$ is an <u>embedding</u> if it is an <u>immersion</u> and $f: M \to f(M) \subset N$ (with subspace top.) is a homeomorphism.

eg:

$\mathbb{R}$  $\xrightarrow{\ \gamma\ }$ 

$|\gamma'| \neq 0$

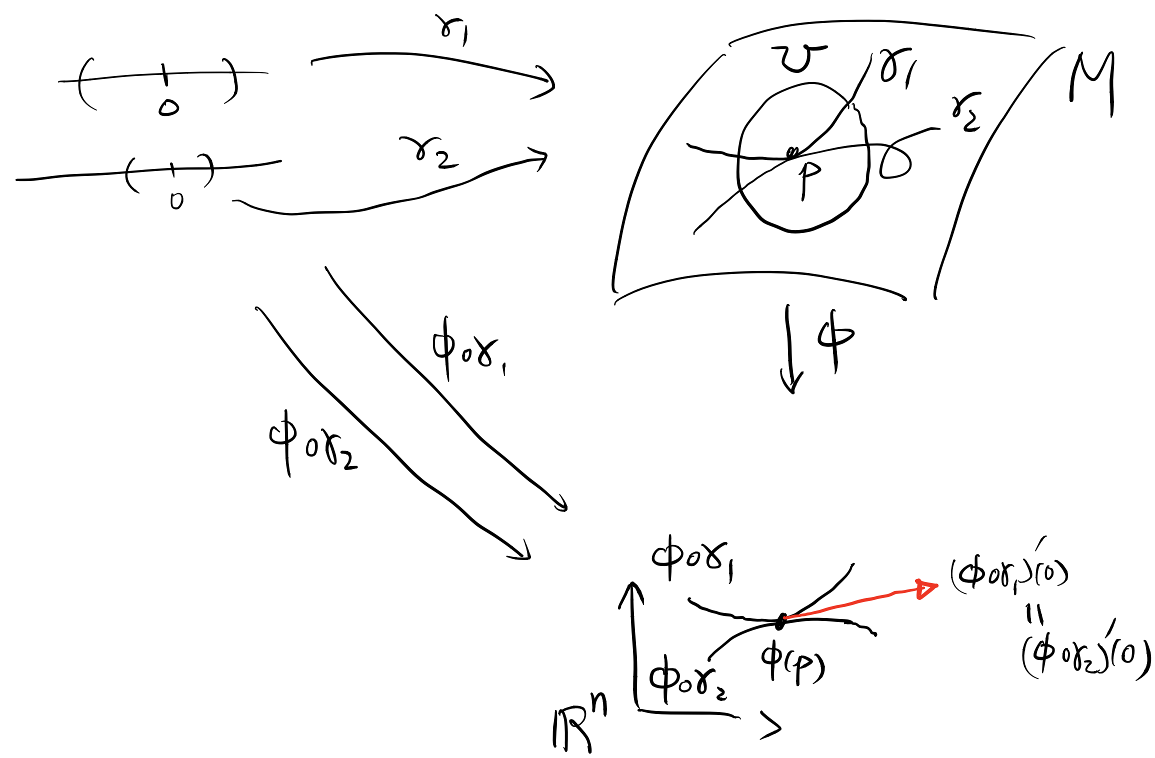
this is an immersion but <u>not</u> embedding (Ex!)

1.3 Tangent vectors

Def 1: Let M be a smooth manifold and $p \in M$. A <u>tangent vector</u> to M at p is an <u>equi. class</u> of C^∞ curves $\gamma: I \to M$, where I = interval containing 0, such that $\gamma(0) = p$, for the equi. relation defined by $\gamma_1 \sim \gamma_2$

$$\Leftrightarrow \quad (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$$

for a chart (U, ϕ) around p.



Ex: Check that the equi. relation is well-defined by showing that for any other chart (V, ψ) around p, we have

$$\boxed{(\psi\circ\gamma)'(0) = D_{\phi(p)}(\psi\circ\phi^{-1})\cdot(\phi\circ\gamma)'(0)}$$

where $D_{\phi(p)}(\psi\circ\phi^{-1})$ is the Jacobi matrix (or differential) of the map $\psi\circ\phi^{-1}$ at $\phi(p)$.

Def 2 (Equivalent definition for tangent vectors)
Let M be a smooth manifold, $p\in M$, (U,ϕ) & (V,ψ) be 2 coordinates charts for M around p. Let u, v be 2 vectors in $\mathbb{R}^n$ (considered as tangent vectors to $\mathbb{R}^n$ at $\phi(p)$ and $\psi(p)$ respectively). We say that

$$\boxed{(U, \phi, u) \sim (V, \psi, v) \iff D_{\phi(p)}(\psi \circ \phi^{-1})\, u = v.}$$

Then a tangent vector to M at p is a equi. class of triples (U, ϕ, u).