Mathematical Analysis III

Tutorial 7 (November 7)

The following were discussed in the tutorial this week:

1. Consider the following two metrics d, ρ on $\mathbb{R}$ given by

$$d(x,y) := |x-y| \quad \text{and} \quad \rho(x,y) := |\tan^{-1} x - \tan^{-1} y|$$

for any $x, y \in \mathbb{R}$. Show that

   (a) a sequence converges in $(\mathbb{R}, d)$ if and only if it converges in $(\mathbb{R}, \rho)$;

   (b) $(\mathbb{R}, d)$ is complete while $(\mathbb{R}, \rho)$ is not.

2. Recall the Contraction Mapping Principle.

Definition. *Let (X,d) be a metric space. A map $T : (X,d) \to (X,d)$ is called a contraction if there is a constant $\gamma \in (0,1)$ such that*

$$d(T(x), T(y)) \leq \gamma d(x,y), \quad \textit{for all } x, y \in X.$$

Theorem. *Let $T : X \to X$ be a contraction in a complete metric space (X,d). Then T has a unique fixed point x_0. Moreover, for any $x \in X$, the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to the fixed point x_0.*

3. Let (X,d) be a compact metric space. Let $f : X \to X$ be a map such that

$$d(f(x), f(y)) < d(x,y) \quad \text{for all } x \neq y.$$

Show that f has a unique fixed point. (Compare it with the Contraction Mapping Principle. Do we have the same conclusion if we just assume that X is a complete metric space?)

(**Hint:** Consider $F(x) := d(x, f(x))$.)

4. Let $X = [-\pi, \pi]$, $f(x) = \cos x$. Show that f satisfies

$$d(f(x), f(y)) < d(x,y) \quad \text{for all } x \neq y$$

but it is not a contraction.