Mathematical Analysis III

Tutorial 6 (October 31)

The following were discussed in the tutorial this week:

1. Recall the definition of a compact set.

2. Show that every closed subset of a compact set is compact.

3. Let A, B be non-empty subsets of a metric space (X, d). Recall

$$d(A, B) := \inf\{d(a, b) : a \in A \text{ and } b \in B\}.$$

   If A is compact and B is closed, show that $A \cap B = \emptyset$ if and only if $d(A, B) > 0$. Does the above statement holds if we only assume that A is closed?

   (This is actually equivalent to Proposition 2.13. I gave a proof using “open cover argument”.)

4. Let (X, d) be a compact metric space and $f : X \to X$ be a map such that

$$d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X.$$

   Show that f is a homeomorphism, that is f is a continuous bijection whose inverse is also continuous. (**Hint:** consider the sequence $\{f^n(x)\}_{n=1}^{\infty}$ for any fixed $x \in X$.)

5. Recall the notion of completeness.

6. Let X, Y be metric spaces such that Y is complete. Let $E \subset X$ and $f : E \to Y$ be a uniformly continuous map. Show that there is a unique uniformly continuous map $F : \overline{E} \to Y$ such that $F|_E = f$.