Solution 5

1. Show that the only open and closed set in $\mathbb{R}$ (with standard metric $d(x,y) = |x-y|$) is the empty set and $\mathbb{R}$ itself.

   Solution. $\mathbb{R}$ itself and hence the empty set are clearly open and closed. Let E be an open and closed set. Suppose E is not empty. Pick a point x_0 from E. Let

   $$a = \inf\{a' : \ (a', x_0] \subset E\}, \quad \text{and} \quad b = \sup\{b' : \ [x_0, b') \subset E\}.$$

   As E is open, we can find some $(a_0, b_0) \subset E$ containing x_0. These two definitions make sense. Suppose a is finite. From the definitions, there exists $\{a_n\} \subset E, a_n \to a$. Since E is closed, one has $a \in E$. As E is also open, $(a-\delta, a+\delta) \subset E$ for some small $\delta > 0$. However, this contradicts with the definition of a. Hence $a = -\infty$. By the same arguments, $b = \infty$, so $(-\infty, \infty) \subset E$, that is $E = \mathbb{R}$.

2. Let $A = \big\{f \in C[-1,1] : \ f(0) = 1, \ f(-1) = -1\big\}$. Show that A is closed in $(C[-1,1], d_\infty)$.

   Solution. Let $f_n \to f$ in $(C[-1,1], d_\infty)$. Then, as $n \to \infty$,

   $$|1 - f(0)| = |f_n(0) - f(0)| \le \max_{x\in[-1,1]} |f_n(x) - f(x)| = \|f_n - f\|_\infty \to 0.$$

   Thus $f(0) = 1$. By the same arguments, one has $f(-1) = -1$. Hence $f \in A$ and A is closed in $(C[-1,1], d_\infty)$.

3. Show that f is continuous from (X,d) to (Y,ρ) if and only if $f^{-1}(F)$ is closed in X for all closed set F in Y.

   Solution. Use $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ to reduce to the statement: f is continuous iff $f^{-1}(G)$ is open for open G.

4. Identify the boundary, interior and closure of the following sets in the indicated metric spaces:

   (a) $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, \text{standard metric})$;

   (b) $B = \displaystyle\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k}\right)$ in $(\mathbb{R}, \text{standard metric})$;

   (c) $C = \mathbb{R}^2 \setminus \{(1/n, 0) : n = 1,2,3,\dots\}$ in $(\mathbb{R}^2, \text{Euclidean metric})$;

   (d) $D = \{f \in C[0,1] : f(0) = f(1)\}$ in $(C[0,1], d_\infty)$.

   Solution.

   (a) $\partial A = [0,1]$, $A^\circ = \emptyset$, $\overline{A} = [0,1]$;

   (b) $\partial B = \{1/k : k = 1,2,3,\dots\} \cup \{0\}$, $B^\circ = B = \bigcup_{k=1}^\infty \left(\frac{1}{k+1}, \frac{1}{k}\right)$, $\overline{B} = [0,1]$;

   (c) $\partial C = \{((0,0),(1,0),(1/2,0),(1/3,0),\dots\}$, $C^\circ = \mathbb{R}^2 \setminus \{((0,0),(1,0),(1/2,0),(1/3,0),\dots\}$, $\overline{C} = \mathbb{R}^2$;

   (d) $\partial D = D$, $D^\circ = \emptyset$, $\overline{D} = D$.
   Let f satisfy $f(0) = f(1)$. For every $\varepsilon > 0$, it is clear we can find some $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \ne g(1)$. It shows that every metric ball $B_\varepsilon(f)$ must contain some functions lying outside this set. Hence, the boundary is also the set itself.

5. Let A and B be subsets of a metric space. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Does $\overline{A \cap B} = \overline{A} \cap \overline{B}$? Justify your answer

   Solution. We have $\overline{C} \subset \overline{D}$ whenever $C \subset D$ from Proposition 2.8. So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Conversely, if $x \in \overline{A \cup B}$, then for any $\varepsilon > 0$, $B_\varepsilon(x)$ either has non-empty intersection with A or B. So there exists $\varepsilon_j \to 0$ such that $B_{\varepsilon_j}(x)$ has nonempty intersection with A or B, so $x \in \overline{A} \cup \overline{B}$. Alternatively, since $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, we have $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ by Proposition 2.8.

   On the other hand, $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is not always true. For instance, consider intervals (a, b) and (b, c). We have $\overline{(a, b)} \cap \overline{(b, c)} = \{b\}$ but $\overline{(a, b) \cap (b, c)} = \emptyset$. Or you take A to be the set of all rationals and B all irrationals. Then $\overline{A \cap B} = \overline{\emptyset} = \emptyset$ but $\overline{A} \cap \overline{B} = \mathbb{R}$.