Solution 3

1. Let f, g and $h \in R[a,b]$. Show that

$$\|f-g\|_2 \le \|f-h\|_2 + \|h-g\|_2 \ .$$

When does the equality sign hold?

Solution. Instead we prove

$$\sqrt{\int (F+G)^2} \le \sqrt{\int F^2} + \sqrt{\int G^2} \ .$$

Taking square, it amounts to proving

$$\left|\int FG\right| \le \sqrt{\int F^2}\sqrt{\int G^2} \ ,$$

which is the Cauchy-Schwarz Inequality we learned in MATH2060.
Equality sign holds if and only if $h = \lambda f + (1-\lambda)g$ almost everywhere, where $\lambda \in [0,1]$.

2. Let $\{\varphi_k\}_{k=1}^{\infty}$, be an orthonormal set on $R[a,b]$. Show that $\forall f \in R[a,b]$,

$$\sum_k < f, \varphi_k >_2^2 \le \int_a^b f^2.$$

(Note that $\{\varphi_k\}_{k=1}^{\infty}$ may not be a basis.)

Solution. It follows from expanding

$$\begin{aligned} 0 &\le \int_a^b \big(f - \sum_{k=1}^n < f, \varphi_k >_2 \varphi_k > \big)^2 \, dx \\ &= \int_a^b f^2 \, dx - 2\sum_{k=1}^n < f, \varphi_k >_2^2 \\ &\quad + \sum_{j,k=1}^n < f, \varphi_k >_2 < f, \varphi_j >_2 < \varphi_j, \varphi_k >_2 \\ &= \int_a^b f^2 \, dx - \sum_{k=1}^n < f, \varphi_k >_2^2 \end{aligned}$$

and then letting n go to ∞.
This is called Bessel inequality.

3. Let f, g be 2π periodic functions integrable on $[-\pi, \pi]$. Show that

$$\int_{-\pi}^{\pi} fg = 2\pi a_0(f)a_0(g) + \pi \sum_{n=1}^{\infty} (a_n(f)a_n(g) + b_n(f))b_n(g)),$$

where a_0, a_n and b_n are corresponding Fourier coefficients.

Solution. Parseval Identity asserts

$$\|f \pm g\|_2^2 = 2\pi(a_0(f) \pm a_0(g))^2 + \pi \sum_{n=1}^{\infty} \Big((a_n(f) \pm a_n(g))^2 + (b_n(f) \pm b_n(g))^2 \Big) \ .$$

The desired result comes from adding up these two identities and dividing by 4.

The meaning. Recall $R_{2\pi}$ is an inner product space with the L^2-product. Let $\mathcal{C}^2$ be the space of $\{c_{-n} = \overline{c_n} \in \mathcal{C} : \ c_n = a_n + ib_n\}$ where all bisequences satisfy $\sum_n |c_n|^2 < \infty$. We can put an inner product on $\mathcal{C}^2$ by setting, in apparent notations,

$$\langle c_n, c_n' \rangle = 2\pi a_0 c_0 + \pi \sum_{n=1}^{\infty} (a_n c_n + b_n d_n).$$

So both $R_{2\pi}$ and $\mathcal{C}^2$ become inner product spaces. The identity above shows that the Fourier transform satisfies

$$\langle f, g \rangle_2 = \langle \hat{f}, \hat{g} \rangle,$$

that is, it is an "isometry".

4. Show that:

(a) $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^4} = \dfrac{\pi^4}{96}$ by Fourier Series of $|x|$

(b) $\sum_{n=0}^{\infty} \dfrac{1}{n^6} = \dfrac{\pi^6}{945}$. by Fourier Series of x^2

Solution.

(a) By the Fourier series of $|x|$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

Then one computes

$$\int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^3}{3}.$$

On the other hand, by the Parseval identity, this equals

$$2\pi \frac{\pi^2}{4} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \pi.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

(b) By the Fourier series of x^2,

$$x^2 = \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$$

Integrate to obtain

$$\frac{x^3}{3} - \frac{\pi^2 x}{3} = -4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx.$$

Then one computes

$$\int_{-\pi}^{\pi} \left(\frac{x^3}{3} - \frac{\pi^2 x}{3} \right)^2 dx = \frac{16\pi^7}{945}.$$

On the other hand, by the Parseval identity, this equals

$$\sum_{n=1}^{\infty}\left(-4\frac{(-1)^{n+1}}{n^3}\right)^2\pi = \sum_{n=1}^{\infty}\frac{16\pi}{n^6}.$$

Hence

$$\sum_{n=0}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

5. Use Wirtinger's inequality to show that $\forall f \in C[0,\pi]$ satisfying $f(0)=f(\pi)=0$ and $f'(x)$ exists for all $x \in [0,\pi]$ and $f' \in R[0,\pi]$, the inequality

$$\int_0^\pi |f|^2 \le \int_0^\pi |f'|^2$$

holds.
When does the equality sign hold?

Solution. Extend the function f as an odd function F on $[-\pi,\pi]$. It is readily checked that F is differentiable on $[-\pi,\pi]$ with $F' \in R[-\pi,\pi]$. The inequality now follows from Wirtinger's inequality. Furthermore, when equality holds, $F = a + b\cos x + c\sin x$. As F is odd, $b=0$. As $F(0)=0, a=0$ and $F = c\sin x$. So F is a scalar multiple of the sine function when equality in this inequality holds.