§4.2 Separability

Def: Let (X, d) be a metric space. A set E in X is dense if $\forall x \in X$, and $\varepsilon > 0$,

$$B_\varepsilon(x) \cap E \neq \emptyset.$$

Notes: (i) Easy to see that E is dense $\Leftrightarrow \overline{E} = X$.

(ii) X is dense (in (X, d)).

eg: If $(X, \text{discrete metric})$, then for $0 < \varepsilon < 1$ & $x \in X$, $B_\varepsilon(x) = \{x\}$. Therefore E is dense in X implies $E = X$. (i.e. X is the only dense set in $(X, \text{discrete})$.)

eg1: In $(\mathbb{R}, \text{standard metric})$, $\mathbb{Q}$ & $\mathbb{R} \setminus \mathbb{Q}$ are dense.

eg2: Let R be a closed and bounded rectangle in $\mathbb{R}^n$, then Weierstrass approximation theorem implies the collection of all polynomials (restricted to R) forms a dense set in $(C(R), d_\infty)$.

(We proved the case $n=1$, and the case for general n follows easily.)

eg3: Also {finite trigonometric series} is dense in $(\mathcal{C}_{2\pi}, d_\infty)$, space of 2π-periodic continuous functions. (See the proof of Prop 1.12).

Def: (i) A metric space X is called a separable space if it admits a countable dense subset.

(ii) A subset is separable if it is separable as a metric subspace.

Prop 4.1 Every subset of a separable space is separable.

Pf: Let (X, d) be separable and E be a countable dense subset of X.

E countable $\Rightarrow$ elements of E can be listed as a sequence $E = \{x_i\}_{i=1}^{\infty}$.

Now, let Y be a subset of X. (Note that x_i may not contained in Y.)

Consider $\forall i \geq 1$, $\forall n \geq 1$ the intersection $Y \cap B_{\frac{1}{n}}(x_i)$.

If $Y \cap B_{\frac{1}{n}}(x_i) \neq \phi$, we can pick a point

$$y_i^n \in Y \cap B_{\frac{1}{n}}(x_i) \subset Y$$

and form a subset

$$F = \{ y_i^n : n \geq 1, i \geq 1 \text{ such that } Y \cap B_{\frac{1}{n}}(x_i) \neq \phi \}.$$

Then F is clearly countable subset of Y.

<u>Claim</u> F is dense in Y.

Pf: $\forall y \in Y$ and $\forall \varepsilon > 0$,

$$\exists x_i \in E \text{ s.t. } d(y, x_i) < \frac{\varepsilon}{4 \cancel{2}}$$

(as E is dense in X)

such n exists, for ε small enough

Therefore, for $\frac{4}{\varepsilon} > n > \frac{2}{\varepsilon}$, $B_{\frac{1}{n}}(x_i) \cap Y \neq \emptyset$

$\Rightarrow \exists y_i^n \in F$ with $y_i^n \in B_{\frac{1}{n}}(x_i) \cap Y$

s.t. $$d(y, y_i^n) \leq d(y, x_i) + d(x_i, y_i^n) < \varepsilon/2 + \tfrac{1}{n} < \varepsilon.$$

$\therefore$ F is dense in Y. #

<u>Prop 4.2</u> Every compact metric space is separable.

To prove this proposition, we need the concept of <u>totally bounded</u>.

<u>Def</u>: A set E in a metric space is called <u>totally bounded</u> if $\forall \varepsilon > 0$, $\exists$ finitely many balls $B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$ such that

$$E \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

<u>Prop 2.11</u> Every (sequentially) compact set is totally bounded.

<u>Pf</u>: Let E be (sequentially) compact. For $\varepsilon > 0$, we pick any $x_1 \in E$ and consider $E \setminus B_\varepsilon(x_1)$. If $E \setminus B_\varepsilon(x_1) = \phi$, we are done. If not, $E \setminus B_\varepsilon(x_1) \neq \phi$ and we can pick a

$x_2 \in E \setminus B_\varepsilon(x_1)$ and consider

$E \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$. If

$E \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2)) = \phi$, then we are done.

If not, we can pick $x_3 \in E \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$

and consider $E \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup B_\varepsilon(x_3))$

$\vdots$

and so on.

If we stop at a finite step, then $\exists\, x_1, \cdots, x_n$

such that $E \setminus \left(\bigcup_{i=1}^{n} B_\varepsilon(x_i) \right) = \phi$. We

are done.

If not, we obtained a sequence

$\{x_1, x_2, x_3, \cdots\}$ such that

$$(*) \begin{cases} d(x_2, x_1) \geq \varepsilon \\ d(x_3, x_i) \geq \varepsilon, \text{ for } i = 1, 2 \\ \quad \vdots \\ d(x_n, x_i) \geq \varepsilon, \text{ for } i = 1, 2, \cdots, n-1 \\ \quad \vdots \end{cases}$$

By (sequential) compactness of E, there exists a subsequence $\{x_{n_j}\}$ and $x \in E$ such that

$$x_{n_j} \to x \quad \text{as } j \to \infty.$$

Then $\exists\, j_0 > 0$ s.t.

$$d(x_{n_j}, x_{n_k}) < d(x_{n_j}, x) + d(x_{n_k}, x)$$
$$< \varepsilon \quad \text{for } n_j, n_k \geq n_{j_0}.$$

We may assume $n_j \geq n_k$, then $(*) \Rightarrow$

$$\varepsilon \leq d(x_{n_j}, x_{n_k}) < \varepsilon$$

which is a contradiction.

So we must stop at a finite step and hence find many point $x_1, \cdots, x_n$ s.t.

$$E \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$ ~~#~~

Remark : A metric space is compact if and only if it is totally bounded and complete. (Ex!)

Proof of Prop 4.2 : By Prop 2.11, every compact set is totally bounded.

$\Rightarrow \forall n, \exists$ finitely many points $x_1^{(n)}, \cdots, x_{N_n}^{(n)}$ such that $\{B_{\frac{1}{n}}(x_i^{(n)})\}_{i=1,\cdots,N_n}$ covers the space.

Then $E = \{x_i^{(n)} : n=1,2,\cdots ; i=1,\cdots,N_n\}$ is a countable set. It is also dense since $\forall x \in X$ & $\forall n \geq 1$, $\exists x_i^{(n)} \in E$ such that

$$d(x, x_i^{(n)}) < \frac{1}{n}. \quad \#$$

<u>eg 4.1</u>: $\mathbb{R}$ is separable as $\mathbb{Q}$ is a countable dense subset. $\mathbb{R}^n$ is separable as $\mathbb{Q}^n$ is a countable dense subset.

By Prop 4.1, all subsets of $\mathbb{R}^n$ are separable.

<u>eg 4.2</u> $(C[a,b], d_\infty)$ is separable.

<u>Pf</u>: Without loss of generality, we may assume

$[a,b]=[0,1]$.

Let $\mathcal{P}=\{$ restriction of polynomials to $[0,1]\}$

and $S=\{p\in\mathcal{P}$: coefficients of $p\in\mathbb{Q}\}$.

Then S is countable. (countable union of countable sets.)

Given a real polynomial $p(x)=\sum_{k=0}^{n}a_k x^k\in\mathcal{P}$,

$(a_k\in\mathbb{R},\ k=0,1,\cdots,n)$ and $\forall\,\varepsilon>0$,

$\exists\ b_k\in\mathbb{Q}$ s.t.

$$|a_k-b_k|<\frac{\varepsilon}{2(n+1)},\ \forall k=0,1,\cdots,n.$$

(Since $\mathbb{Q}$ is dense in $\mathbb{R}$)

Let $q(x)=\sum_{k=0}^{n}b_k x^k\in S$. Then $\forall\, x\in[0,1]$,

$$|p(x)-q(x)|\le\sum_{k=0}^{n}|a_k-b_k|\,x^k<\frac{\varepsilon}{2(n+1)}\cdot(n+1)=\frac{\varepsilon}{2}$$

$\therefore\ \|p-q\|_\infty<\frac{\varepsilon}{2}$.

(In fact, we have proved that S is dense in $\mathcal{P}$)

Since $\mathcal{P}$ is dense in $(C[0,1], d_\infty)$ by Weierstrass theorem, $\forall f \in (C[0,1], d_\infty)$ & $\forall \varepsilon > 0$, $\exists\, p \in \mathcal{P}$ such that $\|f-p\|_\infty < \frac{\varepsilon}{2}$.

Hence $\exists\, q \in \mathcal{S}$ such that

$$\|f-q\|_\infty \le \|f-p\|_\infty + \|p-q\|_\infty < \varepsilon.$$

$\therefore$ $\mathcal{S}$ is a countable dense subset in $(C[0,1], d_\infty)$.

$\therefore$ $(C[0,1], d_\infty)$ is separable. #

<u>Note</u>: A straight forward generation $\Longrightarrow$

$(C(R), d_\infty)$ is separable for R = closed and bounded rectangle in $\mathbb{R}^n$.

<u>Thm 4.3</u> The space $C(X)$ is separable when X is a compact metric space.

[Pf: Omitted. In fact, it needs the Stone-Weierstrass Theorem in next section whose proof will be omitted too.

eg 4.3 (Non-separable Space)

Let $B[a,b] = \{$ bounded functions on $[a,b] \}$ and consider $(B[a,b], d_\infty)$.

One can check that $(B[a,b], d_\infty)$ is a Banach space. (Ex!)

Claim: $(B[a,b], d_\infty)$ is not separable.

Pf: $\forall y \in [a,b]$, define $f_y \in B[a,b]$ by

$$f_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Then $\{ f_y \in B[a,b] : y \in [a,b] \}$ is an uncountable subset in $B[a,b]$ as $f_{y_1} \neq f_{y_2}$ for $y_1 \neq y_2$.

Clearly $d_\infty(f_{y_1}, f_{y_2}) = \sup_{x \in [a,b]} |f_{y_1}(x) - f_{y_2}(x)| = 1$ for $y_1 \neq y_2$.

Hence $B_{\frac{1}{2}}(f_{y_1}) \cap B_{\frac{1}{2}}(f_{y_2}) = \phi$ if $y_1 \neq y_2$.

Now suppose S is a dense subset of $B[a,b]$.

Then $S \cap B_{\frac{1}{2}}(f_y) \neq \phi$, $\forall y \in [a,b]$.

$\Rightarrow \exists\ g_y \in S \cap B_{\frac{1}{2}}(f_y) \subset B[a,b]$.

s.t. $g_{y_1} \neq g_{y_2}$ if $y_1 \neq y_2$

$\Rightarrow \{g_y\}_{y\in[a,b]}$ forms an uncountable subset of S.

Hence S is uncountable.

$\therefore$ $B[a,b]$ has no countable dense subset. $\times$

§4.3 The Stone-Weierstrass Theorem

(optimal reading)

Thm 4.4 (Stone-Weierstrass) Let $\mathcal{A}$ be a subalgebra of $C(X)$ where X is a compact metric space. Then $\mathcal{A}$ is dense in $C(X)$ if and only if it has the separating points and non-vanishing properties.

(Pf : Omitted)

Notes : (i) A subspace $\mathcal{A}$ is called a subalgebra of $C_b(X)$ if it is closed under multiplication of functions.
(pointwise)

(ii) A subalgebra is called to satisfy the separating points property if $\forall\ x_1 \neq x_2 \in X$, $\exists\ f \in \mathcal{A}$ satisfying $f(x_1) \neq f(x_2)$.

(iii) A subalgebra is called to satisfy the non-vanishing property if $\forall\ x \in X$, $\exists\ g \in \mathcal{A}$ such that $g(x) \neq 0$.