Ch4 The Space of Continuous Functions

§4.1 Spaces of Continuous Functions

<u>Notation</u>: For a metric space (X, d), we denote by $C(X)$ the vector space of all continuous functions defined on X; and $C_b(X)$ the vector space of all bounded continuous functions on X.

i.e. $C_b(X) = \{ f \in C(X) : |f(x)| \le M, \forall x \in X, \text{ for some } M \}$

<u>Recall that</u>: A norm $\|\cdot\|$ on a real vector space X is defined by the following properties:

(N1) $\|x\| \ge 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad (\alpha \in \mathbb{R})$

(N3) $\|x+y\| \le \|x\| + \|y\|$.

And a vector space with norm $(X, \|\cdot\|)$ is called a norm space. A norm space has a natural metric $d(x,y) = \|x-y\|$.

<u>Fact</u>: $\|f\|_\infty = \sup_{x \in X} |f(x)|$ is a norm on $C_b(X)$.

(Pf: Ex!)

$\therefore$ $(C_b(X), \|\cdot\|_\infty)$ is a norm space and has metric $d_\infty(f,g) = \|f-g\|_\infty$.

And $(C_b(X), d_\infty)$ is a metric space.

Prop: $(C_b(X), d_\infty)$ is complete. (for any metric space (X,d))

Pf: Let $\{f_n\}$ be a Cauchy seq. in $(C_b(X), d_\infty)$

Then $\forall \varepsilon>0$, $\exists n_0 \geq 0$ s.t.

$$\|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \quad \forall m,n \geq n_0.$$

In particular, $\forall x \in X$,

$$(*)_1: \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \frac{\varepsilon}{4}, \quad \forall m,n \geq n_0$$

$\Rightarrow$ $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$.

By completeness of $\mathbb{R}$ (not X), $\lim_{n\to\infty} f_n(x)$ exists and, in general, depends on x. Let denote it by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \forall x \in X.$$

This gives a function f on X.

Claim(1). f is bounded.

Pf: Letting $m \to \infty$ in $(*)_1$, we have

$\forall \varepsilon > 0$, and $\forall x \in X$,

$$(*)_2: \qquad |f(x) - f_n(x)| \le \frac{\varepsilon}{4}, \ \forall n \ge n_0$$

In particular, $|f(x) - f_{n_0}(x)| \le \frac{\varepsilon}{4}$, $\forall \varepsilon > 0$, $\forall x \in X$.

$\Rightarrow \forall x \in X$, $|f(x)| \le \frac{\varepsilon}{4} + |f_{n_0}(x)| \le \frac{\varepsilon}{4} + M_0$,

where M_0 is a bound for f_{n_0}.

$\therefore$ f is bounded.

Claim (2): f is continuous.

Pf: f_{n_0} cts $\Rightarrow$ $\forall x_0 \in X$ & $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{4}$, $\forall d(x, x_0) < \delta$.

Then together with $(*)_2$,

$$|f(x) - f(x_0)| \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$$\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon, \ \forall d(x, x_0) < \delta.$$

$\therefore$ f is cts at x_0.

Since $x_0 \in X$ is arbitrary, f is cts on X.

Claims (1) & (2) $\Rightarrow$ $f \in C_b(X)$.

Finally, by $(*)_2$, $\sup_{x\in X} |f(x) - f_n(x)| \le \frac{\varepsilon}{4}$, $\forall n \ge n_0$

i.e. $d_\infty(f_n, f) \le \frac{\varepsilon}{4}$, $\forall n \ge n_0$

So $d_\infty(f_n, f) \to 0$ as $n \to \infty$

That is $f_n \to f$ in $(C_b(X), d_\infty)$. #

Notes: (i) If (X, d) is compact, then $C(X) = C_b(X)$.
(Ex!)

(ii) In functional analysis, a complete normed vector space is called a Banach space. So $(C_b(X), d_\infty)$ is a Banach space.

(iii) We usually just write $C_b(X)$ for $(C_b(X), d_\infty)$ if no confusion.

(iv) $C_b(X)$ is usually of infinite dimensional:

egs: When $X = \mathbb{R}^n$ a subset with non-empty interior in $\mathbb{R}^n$.

Explicit eg: $X = [0,1] \subset \mathbb{R}$, then $\{x^n\}_{n=0}^{\infty} \subset C_b(X)$

Clearly, $\{x^n\}_{n=0}^{\infty}$ is a linearly indep. subset.

$\Rightarrow$ $C_b(X)$ is of infinite dimensional.

But $C_b(X)$ could be of finite dimension:

eg: $X=\{p_1,\cdots,p_n\}$ finite set with discrete metric

Then
$$\begin{array}{ccc} X & \to & \mathbb{R}^n \\ \Psi & & \Psi \\ f & \mapsto & (f(p_1),\cdots,f(p_n)) \end{array}$$
is a linear bijection.

(v) In general, $C(X)$ may contain unbounded function and sup-norm doesn't define. However, in some cases, we still possible to define a metrix on $C(X)$:

eg $X=\mathbb{R}^n$, $\overline{B_n(0)}=\{|x|\leq n\}$, $\forall n=1,2,3,\cdots$

$\forall f\in C(\mathbb{R}^n)$, define
$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|f-g\|_{\infty,\overline{B_n(0)}}}{1+\|f-g\|_{\infty,\overline{B_n(0)}}}.$$

Then d is a complete metric on $C(\mathbb{R}^n)$.