<u>Remark</u>: Sometime one will need a quantitative estimates for ρ_0 and R:

Let $M_{DF}(\rho) = \sup_{x \in B_\rho(0)} \| DF(x) - DF(0) \|$ be the <u>modulus of continuity</u> of DF at 0. Then

$M_{DF}(\rho) \downarrow 0$ as $\rho \to 0$ (as DF cts). And

ρ_0 & R can be chosen by requiring

$$\boxed{M_{DF}(\rho_0) \le \frac{1}{2\|L^{-1}\|} \quad \& \quad R \le \frac{\rho_0}{2\|L^{-1}\|}.}$$

<u>eg 3.8</u>: Consider $\begin{cases} x - y^2 = a \\ x^2 + y + y^3 = b \end{cases}$

Let $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y^2 \\ x^2 + y + y^3 \end{pmatrix}$.

Then $F\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ &

$$DF\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & -2y \\ 2x & 1 + 3y^2 \end{pmatrix}$$

$\therefore L = DF(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is invertible & $L^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \quad \|L^{-1}\| = \sqrt{2}$.

Moreover $\left\|DF\left(\binom{x}{y}\right) - DF(0)\right\|$

$$= \left\| \begin{pmatrix} 0 & -2y \\ 2x & 3y^2 \end{pmatrix} \right\| = \sqrt{4x^2+4y^2+9y^4}$$

To find $\rho_0 > 0$, we want

$$M_{DF}(\rho_0) \leq \frac{1}{2\|L^{-1}\|}$$

i.e. need $\displaystyle\sup_{\binom{x}{y}\in B_{\rho_0}(0)} \sqrt{4x^2+4y^2+9y^4} \leq \frac{1}{2\sqrt{2}}$

Using polar coordinate $\sqrt{4x^2+4y^2+9y^4} = \sqrt{4\rho^2+9\rho^4\sin^4\theta}$

$\forall \ \binom{x}{y} = \binom{\rho\cos\theta}{\rho\sin\theta}$ $\rho<\rho_0$

we see $\quad M_{DF}(\rho_0) = \sqrt{4\rho_0^2+9\rho_0^4}$.

So we can choose ρ_0 by

$$\sqrt{4\rho_0^2+9\rho_0^4} = \frac{1}{2\sqrt{2}}$$

$\Rightarrow \quad \ldots \quad \rho_0 = \dfrac{\sqrt{\sqrt{82}-8}}{6} \sim 0.17$ (Check!)

Then take $R = \dfrac{\rho_0}{2\|L^{-1}\|} = \dfrac{1}{2\sqrt{2}} \dfrac{\sqrt{\sqrt{82}-8}}{6} \sim 0.06$

$\therefore \forall \binom{a}{b} \in B_{\sim 0.06}(0)$, the system is solvable in

$\overline{B_{\sim 0.17}(0)}$.

<u>Def:</u> A C^k-map $F: V \to W$ (V, W open in $\mathbb{R}^n$) is a <u>C^k-diffeomorphism</u> if F^{-1} exists and is also C^k.

<u>Note</u>: (i) The IFT can be rephased as:

If $F: U \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}^n \in C^k$, and DF is nonsingular at a point $p_0 \in U$, then F is a C^k-diffeomorphism between some nbds V and W of p_0 & $F(p_0)$ respectively.

(ii) If $F: V \to W$ is a C^k-diffeomorphism, then $\forall$ function $\varphi: W \to \mathbb{R}$, there corresponds a function $\psi = \varphi \circ F: V \to \mathbb{R}$. Conversely, $\forall$ function $\psi: V \to \mathbb{R}$, there corresponds a function $\varphi = \psi \circ F^{-1}: W \to \mathbb{R}$. Moreover, φ is $C^k \iff \psi$ is C^k. Thus every

C^k- diffeomaphism gives rise to a "<u>local change of coadinates</u>".