Def: Let (X, d) be a metric space.
Then for any non-empty $Y \subset X$,
$(Y, d|_{Y\times Y})$ is called a metric subspace
of (X, d)

Notes: (i) metric subspace is a metric space.
(ii) We simple write (Y, d) for $(Y, d|_{Y\times Y})$.
(iii) A metric subspace of a normed space needs not be a normed space, only if the subset is also a vector subspace.

Def: A sequence $\{x_n\}$ in a metric space (X, d) is said to be converge to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

In this case, we write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ in X.

Prop (Uniqueness of limit)
If $x_n \to x$ & $x_n \to y$ in a metric space, then $x = y$.

egs: (i) Convergence in $(\mathbb{R}^n, d_2)$ is the usual convergence in adv. calculus.

(ii) Convergence in $(C[a,b], d_\infty)$ is the uniform convergence of seq. of functions in $C[a,b]$.

<u>Def</u>: Let d and ρ be 2 metrics defined on X.

(1) We call ρ is <u>stronger than</u> d or d is <u>weaker than</u> ρ, if $\exists\, C>0$ s.t.

$$d(x,y) \le C\rho(x,y), \quad \forall\, x,y \in X.$$

(2) They are <u>equivalent</u> if ρ is stronger and weaker than d. i.e. $\exists\, C_1, C_2 > 0$ s.t.

$$d(x,y) \le C_1\rho(x,y) \le C_2 d(x,y), \quad \forall\, x,y \in X.$$

$$\left(\text{or} \quad c_1 d(x,y) \le \rho(x,y) \le c_2 d(x,y), \quad \forall x,y \in X\right)$$

<u>Prop</u>: (1) If ρ is stronger than d, then $\{x_n\}$ converges in (X,ρ) implies $\{x_n\}$ converges in (X,d), and hence the same limit.

(2) If ρ is equivalent to d, then $\{x_n\}$ converges in (X,ρ) if and only if $\{x_n\}$ converges in (X,d).

(3) "equivalent" of metrics defined above is an equivalent relation.

eg: On $\mathbb{R}^n$, $\begin{cases} d_1(x,y) = \sum_i |x_i - y_i| \\ d_2(x,y) = \left(\sum_i |x_i - y_i|^2\right)^{1/2} \\ d_\infty(x,y) = \max_i |x_i - y_i| \end{cases}$

Check: (i) $d_2(x,y) \le \sqrt{n}\, d_\infty(x,y) \le \sqrt{n}\, d_2(x,y)$

(ii) $d_1(x,y) \le n\, d_\infty(x,y) \le n\, d_1(x,y)$.

Therefore, d_1, d_2, & d_∞ are equivalent metrics on $\mathbb{R}^n$.

eg: $X = C[a,b]$, $\begin{cases} d_1(f,g) = \int_a^b |f-g| \\ d_\infty(f,g) = \max_{[a,b]} |f-g| \end{cases}$

Then clearly

$$d_1(f,g) \le (b-a)\, d_\infty(f,g),\ \forall f,g \in C[a,b].$$

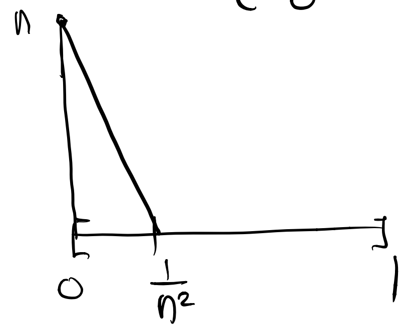
$\therefore$ d_∞ is stronger than d_1.

However, it is impossible to find $C>0$ s.t.

$$d_\infty(f,g) \le C\, d_1(f,g),\ \forall f,g \in C[a,b].$$

Pf: Define f_n on $[a,b] = [0,1]$

$$f_n(x) = \begin{cases} -n^3 x + n, & x \in [0, \frac{1}{n^2}] \\ 0, & x \in (\frac{1}{n^2}, 1] \end{cases}$$



Then $d_1(f_n, 0) = \int_0^1 |f_n| = \frac{1}{2n} \to 0$

& $d_\infty(f_n, 0) = \max_{[0,1]} |f_n(x)| = n$

$\therefore \quad n = d_\infty(f_n, 0) \le C' d_1(f_n, 0) = \frac{C}{2n}, \ \forall n$

which is impossible.

$\therefore \quad d_1$ is <u>not</u> stronger than d_∞.

Therefore d_1 & d_∞ are <u>not</u> equivalent.

<u>Def</u>: Let $f : (X, d) \to (Y, \rho)$ be a mapping between 2 metric spaces, and $x \in X$. We call f is <u>continuity at</u> x if

$f(x_n) \to f(x)$ in (Y, ρ) whenever $x_n \to x$ in (X, d).

It is <u>continuous</u> on a set $E \subset X$ if it is continuous at every point of E.