Cor 1.19 (Wirtinger's Inequality) (a Poincaré inequality)

For any 2π-periodic (real) function f integrable on $[-\pi,\pi]$ s.t. f' is also integrable on $[-\pi,\pi]$, we have

$$\int_{-\pi}^{\pi} (f(x)-a_0)^2\,dx \leq \int_{-\pi}^{\pi} (f'(x))^2\,dx$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f$, and equality holds if and only if $f(x) = a_0 + a_1\cos x + b_1 \sin x$.

Pf: $f(x) - a_0 = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x)$ (since f' exists)

By Parseval's Identity

$$\int_{-\pi}^{\pi} (f(x)-a_0)^2\,dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Similarly
$$\int_{-\pi}^{\pi} \left(f' - \overset{0}{\cancel{a_0(f')}}\right)^2 = \pi \sum_{n=1}^{\infty} \left(a_n(f')^2 + b_n(f')^2\right)$$

$$\Rightarrow \int_{-\pi}^{\pi} (f')^2 = \pi \sum_{n=1}^{\infty} \left[(nb_n)^2 + (-na_n)^2\right]$$

$$= \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2)$$

$$\int_{-\pi}^{\pi} f'^2 - \int_{-\pi}^{\pi} (f-a_0)^2 \geq \pi \sum_{n=1}^{\infty} (n^2-1)(a_n^2+b_n^2) \geq 0.$$

If "equality" holds, then $0 = \sum_{n=1}^{\infty} (n^2-1)(a_n^2+b_n^2)$

$\Rightarrow \quad a_n = b_n = 0 \quad \forall\ n \geq 2$

$\therefore \quad f = a_0 + a_1 \cos x + b_1 \sin x$ #

<u>Note</u>: If we denote

$R_{2\pi} = \{ 2\pi\text{-periodic (real) functions integrable on } [-\pi,\pi] \}$

$\mathcal{C} = \left\{ \text{cpx bisequence } \{c_n\} \text{ with } c_n \to 0 \ \& \ c_{-n} = \overline{c_n} \right\}$

Then $\mathcal{F} : R_{2\pi} \to \mathcal{C}$

$f \mapsto \{c_n(f)\}$

defines a map from $R_{2\pi}$ to $\mathcal{C}$.

Cor 1.17 $\Rightarrow$ $\mathcal{F}$ is "essentially" one-to-one

i.e. $\mathcal{F}(f_1) = \mathcal{F}(f_2) \Leftrightarrow f_1 = f_2$ almost everywhere.

And it is easy to see

(1) $\mathcal{F}$ is linear

(2) If $f \in R_{2\pi}$ is k-th differentiable & $f^{(k)} \in R_{2\pi}$

then $\boxed{c_n\left(f^{(k)}\right) = (in)^k c_n(f), \quad \forall n \in \mathbb{Z}}$

(3) For $f \in R_{2\pi}$ & $a \in \mathbb{R}$, defined $f_a \in R_{2\pi}$ by $f_a(x) = f(x+a)$, $\forall a \in \mathbb{R}$.

Then $\boxed{c_n(f_a) = e^{ina} c_n(f), \ \forall n \in \mathbb{Z}}$

(Pf: easy exercise)

§1.6 The Isoperimetric Problem

Recall: For a domain D (in $\mathbb{R}^2$) enclosed by a simple closed curve $\gamma = (x(t), y(t))$, $t \in [a,b]$, we have

Green's Thm $\displaystyle\int_\gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

for C^1 functions P & Q on $\bar{D}$ $(= D \cup \gamma)$

Taking $P \equiv 0$, $Q \equiv x$, Green's Thm $\Rightarrow$

$$\int_\gamma x\,dy = \iint_D 1\,dx\,dy = \text{Area}(D).$$

$$\therefore \boxed{\text{Area}(D) = \int_\gamma x\,dy = \int_a^b x(t)\,y'(t)\,dt}$$

Suppose that γ is parametrized by arc-length i.e. $|\gamma'|^2 = x'(t)^2 + y'(t)^2 = 1, \; \forall t \in [a,b]$.

By scaling, we can assume γ has length 2π:

$$L(\gamma) = \int_0^{2\pi} \underbrace{\sqrt{x'^2+y'^2}}_{\substack{\shortparallel \\ 1}} \, ds = 2\pi,$$

$s \in [0, 2\pi]$ is the arc-length parameter.

Then
$$\text{Area}(D) = \int_0^{2\pi} x(s)\, y'(s)\, ds$$

$$= \int_{-\pi}^{\pi} x(s)\, y'(s)\, ds \quad \left(\begin{array}{l}\text{as } \gamma \text{ closed curve} \\ \Leftrightarrow x, y \text{ } 2\pi\text{-periodic}\end{array}\right)$$

$$= \int_{-\pi}^{\pi} [x(s) - a_0]\, y'(s)\, ds \quad \text{where } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x$$

$$\leq \frac{1}{2} \int_{-\pi}^{\pi} 2\, |x(s) - a_0|\, |y'(s)|\, ds$$

$$\leq \frac{1}{2} \int_{-\pi}^{\pi} \left[(x(s) - a_0)^2 + y'^2(s) \right] ds$$

Wirtinger's Inequality

$$\leq \frac{1}{2} \int_{-\pi}^{\pi} x'^2(s)\, ds + \frac{1}{2} \int_{-\pi}^{\pi} y'^2(s)\, ds$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \underbrace{(x'^2 + y'^2)}_{=1}\, ds$$

$$= \pi$$

Note that $\pi = \text{Area}(B_1(0))$ & $L(\partial B_1(0)) = 2\pi$

$\therefore$ By scaling back, among all the simple closed C^1-curves of the same parameter, the circle encloses the largest area.

<u>Uniqueness</u> If γ of $L(\gamma)=2\pi$ is such that $\text{Area}(D)=\pi$. Then all the inequalities above become equalities. In particular, equality case of Wirtinger's ineq

$$\Rightarrow \quad x(s) = a_0 + a_1\cos s + b_1 \sin s$$
$$= a_0 + r\cos(s-x_0)$$

where $\begin{cases} r=\sqrt{a_1^2+b_1^2} > 0 \\ \cos x_0 = \frac{a_1}{r} \end{cases}$ $\leftarrow$ (otherwise $x(s)\equiv a_0$. Then $x'^2+y'^2=1$ $\Rightarrow y'^2=1$ $\Rightarrow y=\pm s+b$ cannot be 2π-periodic.)

The inequality $2ab \le a^2+b^2$ becomes $2ab=a^2+b^2 \Rightarrow a=b$

$$\therefore \quad x(s)-a_0 = y'(s)$$
$$\Rightarrow \quad y'(s) = r\cos(s-x_0)$$
$$\Rightarrow \quad y(s) = r\sin(s-x_0)+b_0, \quad b_0=\text{const}$$

$\therefore$ $\gamma=(a_0+r\cos(s-x_0),\ b_0+r\sin(s-x_0))$ is a circle.

In conclusion

Thm 1.20 Among all closed simple C^1-curves of the same parameter, **only** the circle encloses the largest area.

Ch2 Metric Space

In this chapter, $\mathbb{X}$ always denotes a non-empty set.

Def: A metric on $\mathbb{X}$ is a function

$$d : \mathbb{X} \times \mathbb{X} \to [0, +\infty) \text{ such that}$$

$\forall\, x, y, z \in \mathbb{X}$

(M1) $d(x,y) \geq 0$ & "equality holds $\Leftrightarrow x = y$".

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,z) + d(z,x)$

The pair $(\mathbb{X}, d)$ is called a metric space.

Note: Condition (M3) is called the triangle inequality.

Def: Let $(\mathbb{X}, d)$ be a metric space. The metric ball of radius r centered at x

or simple the ball $B_r(x) = \{ y \in \mathbb{X} : d(y,x) < r \}$

eg 2.1 $(\mathbb{X} = \mathbb{R},\ d(x,y) = |x - y|)$ is a metric space.

eg 2.2 Let $\mathbb{X} = \mathbb{R}^n$, $d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

(Euclidean metric)

for $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)\in\mathbb{R}^2$.

Then $(\mathbb{R}^n, d_2)$ is a metric space.

Recall the proof: $\|x\|^2=\sum_{i=1}^n x_i^2$

Then $\|x+y\|^2=\langle x+y, x+y\rangle=\|x\|^2+2\langle x,y\rangle+\|y\|^2$

By Cauchy-Schwarz inequality

$$|\langle x,y\rangle|\le\|x\|\|y\|$$

$$\Rightarrow \quad \|x+y\|^2\le(\|x\|+\|y\|)^2$$

$$\Rightarrow \quad \|x+y\|\le\|x\|+\|y\|, \quad \forall x,y\in\mathbb{R}^n$$

Replace x by $x-z$
y by $z-y$,

then $\|x-y\|\le\|x-z\|+\|z-y\|$.

eg 2.3 Let $X=\mathbb{R}^n$, $d_1(x,y)=\sum_{i=1}^n|x_i-y_i|$

$$d_\infty(x,y)=\max_{i=1,\cdots,n}|x_i-y_i|$$

Then $(\mathbb{R}^n,d_1)$ & $(\mathbb{R}^n,d_\infty)$ are metric spaces.

Generalization of egs 2.2 & 2.3 to function space:

eg 2.4 Let $C[a,b]=\{$(real) continuous functions on $[a,b]\}$

$\forall f, g \in C[a,b]$, define

$$d_\infty(f,g) = \|f-g\|_\infty = \max\{|f(x)-g(x)| : x \in [a,b]\}$$

Then $(C[a,b], d_\infty)$ is a metric space (Ex!)

Similary, one can define

$$d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx.$$

It is also easy to verify that $(C[a,b], d_1)$ is a metric space.

The natural generalization of the Euclidean metric to $C[a,b]$ is

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2}.$$

(M1) & (M2) are clear for d_2 (as f, g cts)

To see (M3), note that $d_2(f,g) = \|f-g\|_2$

Question 1 in HW3 $\Rightarrow$ d_2 satisfies (M3).

$\therefore$ $(C[a,b], d_2)$ is a metric space.

eg 2.5 On $X = R[a,b] = \left\{ \begin{array}{l} \text{Riemann integrable functions} \\ \text{on } [a,b] \end{array} \right\}$

d_1 is still defined $d_1(f,g) = \int_a^b |f-g|$

However, (M1) doesn't satisfied as

$$d_1(f,g)=0 \Leftrightarrow f=g \text{ almost everywhere}$$
$$\not\Rightarrow f=g.$$

$\therefore$ d_1 is <u>not</u> a metric on $R[a,b]$.

To overcome this, we consider $X = R[a,b]/\sim$

where "$\sim$" is an <u>equivalent</u> relation on $R[a,b]$

defined by $\quad f \sim g \Leftrightarrow f=g$ almost everywhere.

(check: "$\sim$" is an equivalent relation.)

Then elements of $R[a,b]/\sim$ can be represented as

$$[f] \text{ or } \bar{f} = \{ g \in R[a,b] : g = f \text{ almost everywhere} \}$$

Now define $\tilde{d}_1$ on $R[a,b]/\sim$ by

$$\tilde{d}_1(\bar{f},\bar{g}) = d_1(f,g)$$

<u>Check</u>: $\tilde{d}_1$ is well-defined, i.e. indep. of the choice of representitivitives f & g:

$\forall f_1 \in \bar{f}, g_1 \in \bar{g}$. Then

$$d_1(f_1,g_1) = \int |f_1-g_1| \le \int |\cancelto{0}{f_1-f}| + \int |f-g| + \int |\cancelto{0}{g-g_1}|$$
$$= d_1(f,g)$$

Similarly $d_1(f,g) \leq d_1(f_1,g_1)$

$\therefore \quad d_1(f,g) = d_1(f_1,g_1)$.

Then it is straigh forward to verify that

$\left(R[a,b]/\sim , \widehat{d_1} \right)$ is a metric space.

Similarly for $\left(R[a,b]/\sim , \widehat{d_2} \right)$ is a metric space

& note that $\widetilde{d_2}$ is the L^2-distance we defined before.

<u>Def</u>: A <u>norm</u> $\|\cdot\|$ is a function on a real vector space X to $[0,+\infty)$ s.t. $\forall x,y \in X$, & $\alpha \in \mathbb{R}$

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$"

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$

(N3) $\|x+y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a <u>normed space</u>.

And $d(x,y) \overset{\text{def}}{=} \|x-y\|$ is called the <u>metric induced</u> by the norm $\|\cdot\|$.

(Ex: Show that $d(x,y) = \|x-y\|$ is a metric with the property $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$, $\forall \alpha \in \mathbb{R}$)

egs : $\|X\|_2 = (\sum x_i^2)^{1/2}$, $\|X\|_1 = \sum |x_i|$,

$\|X\|_\infty = \max\{|x_1|, \cdots, |x_n|\}$

are norms on $\mathbb{R}^n$

$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$, $\|f\|_1 = \int_a^b |f|$,

$\|f\|_\infty = \max\{|f(x)| : x \in [a,b]\}$

are norms on $C[a,b]$.

We've seen "norm" $\xrightarrow{\text{induces}}$ "metric"

But not all metric induced from norm.

eg : X = non-empty set.

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$ discrete metric on X.

- X not necessary a vector space, so d not induced by norm.
- Even for vector space :

  $\left.\begin{matrix}1\\0\end{matrix}\right\} = d(\alpha x, \alpha y) = |\alpha|\, d(x,y) = \left\{\begin{matrix}|\alpha|\\0\end{matrix}\right.$

  Contradiction for $|\alpha| \neq 1$. (for $x \neq y$).