Cor 1.15 For 2π-periodic function f integrable on $[-\pi,\pi]$ and $n \geq 1$,

$$\|f - S_n f\|_2 \leq \|f - g\|_2 \quad \forall\, g \text{ of the form}$$

$$g = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$$

with $\alpha_0, \alpha_k, \beta_k \in \mathbb{R}$

Pf: By def. of Fourier coefficients $S_n f = P_n f$ of the span $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos kx, \frac{1}{\sqrt{\pi}}\sin kx\right\}_{k=1}^{n}$

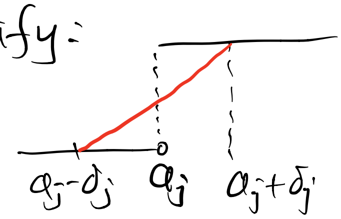
Thm 1.16 For 2π-periodic (real) function f integrable on $[-\pi,\pi]$,

$$\lim_{n\to\infty} \|S_n f - f\|_2 = 0 .$$

i.e. the n-partial sum of the Fourier series of f converges to f in L^2-sense.

Pf: Step 1: $\forall\, \varepsilon > 0$, $\exists$ a 2π-periodic Lip. cts function g s.t. $\|f - g\|_2 < \varepsilon/2$.

(Ex: Hint: find step function approximating f as before, then modify:



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Step 2: Completion of the proof.

Applying Thm 1.7 to the function g in Step 1:

$\exists N > 0$ s.t. $\|g - S_N g\|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}$

uniform convergence

Thus $\|g - S_N g\|_2 = \sqrt{\int_{-\pi}^{\pi} (g - S_N g)^2} \le \sqrt{2\pi}\, \|g - S_N g\|_\infty$

$= \frac{\varepsilon}{2}$.

By Cor 1.15,

$\|f - S_N f\|_2 \le \|f - S_N g\|_2$ ($S_N g$ is of the form $\alpha_0 + \sum_{k=1}^{N} \alpha_k \cos kx + \beta_k \sin kx$)

$\le \|f - g\|_2 + \|g - S_N g\|_2$ (Ex!)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

(step 1)

Finally, since $E_N \subset E_n$, $\forall n \ge N$
(have more generators),

we have $\forall n \ge N$,

$\|f - S_n f\|_2 \le \|f - S_N f\|_2$ (by Cor 1.15 over the subsp. E_n)

$< \varepsilon$.

$\therefore \quad \lim_{n\to\infty} \|S_n f - f\|_2 = 0 \quad$ ※

Eg 1.17 (a) Suppose that f_1 & f_2 are 2π-periodic integrable functions on $[-\pi,\pi]$ with the same Fourier series. Then $f_1 = f_2$ almost everywhere (i.e. $f_1 = f_2$ except a set of measure zero.)

(b) Suppose that f_1 and f_2 are 2π-periodic continuous functions with the same Fourier series. Then $f_1 = f_2$.

Recall: A set E is said to be of measure zero if $\forall\, \varepsilon > 0$, $\exists$ countably many intervals $\{I_k\}$ s.t.

$$E \subset \bigcup_k I_k \quad \&$$

$$\sum_k |I_k| < \varepsilon .$$

Pf: (a) Let $f = f_1 - f_2$, then $a_n(f) = b_n(f) = 0$

$$\Rightarrow \quad S_n f = 0 \quad \forall\, n \geq 0$$

Hence $\quad \lim_{n\to\infty} \|S_n f - f\|_2 = 0$

$$\Rightarrow \quad \|f\|_2 = 0$$

By theory of Riemann integral, $f = 0$ almost everywhere.

(b) We still have $\|f\|_2 = 0$. As f_1, f_2 cts

$\Rightarrow f^2$ cts $\geq 0 \Rightarrow f^2 \equiv 0$. ✗

Corl.18 (Parseval's Identity)

For every 2π-periodic function f integrable on $[-\pi, \pi]$

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where a_0, a_n, b_n are Fourier coefficients of f.

Pf: By def. of a_n:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx \Rightarrow \sqrt{2\pi}\, a_0 = \langle f, \tfrac{1}{\sqrt{2\pi}} \rangle_2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \Rightarrow \sqrt{\pi}\, a_n = \langle f, \tfrac{1}{\sqrt{\pi}}\cos nx \rangle_2$$

$n \geq 1$.

Similarly $\sqrt{\pi}\, b_n = \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle_2$, $n \geq 1$.

Then $\langle f, S_N f \rangle_2 = \langle (\underbrace{f - S_N f}_{\text{orthogonal to the subsp}}) + S_N f, S_N f \rangle_2$

(by Corl.15)

$$= \langle S_N f, S_N f \rangle_2$$

$$= \int_{-\pi}^{\pi} \left(a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx\right)^2 dx$$

$$= 2\pi a_0^2 + \sum_{k=1}^{N} (\pi a_k^2 + \pi b_k^2)$$

Hence

$$0 \overset{\text{Thm 1.16}}{=} \lim_{N\to 0} \|f - S_N f\|_2^2$$

$$= \lim_{N\to 0} \left(\|f\|_2^2 - 2\langle f, S_N f\rangle_2 + \|S_N f\|_2^2 \right)$$

$$= \lim_{N\to 0} \left(\|f\|_2^2 - 2\|S_N f\|_2^2 + \|S_N f\|_2^2 \right)$$

$$= \lim_{N\to 0} \left(\|f\|_2^2 - \|S_N f\|_2^2 \right)$$

$$\therefore \|f\|_2^2 = \lim_{N\to 0} \left[2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2) \right]$$

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eg: Recall $f_1(x) = x$ on $[-\pi, \pi]$ has Fourier series

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$\Rightarrow \int_{-\pi}^{\pi} x^2 \, dx = \int_{-\pi}^{\pi} f_1^2 = \pi \sum_{n=1}^{\infty} \frac{4}{n^2}$$

(check) $\Rightarrow$ $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$ (Euler famula)