Fourier Series of 2T-periodic (real) functions

Let f be a 2T-periodic function

Then $g(x) = f(\frac{T}{\pi}x)$ is 2π-periodic

Therefore

$$f\left(\frac{T}{\pi}x\right) = g(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with
$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{T}\int_{-T}^{T} f(y)\cos\left(\frac{n\pi}{T}y\right)dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx = \frac{1}{T}\int_{-T}^{T} f(y)\sin\left(\frac{n\pi}{T}y\right)dy. \end{cases}$$

Hence

$$f(y) \sim a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{T}y\right) + b_n \sin\left(\frac{n\pi}{T}y\right)\right]$$

with
$$\begin{cases} a_0 = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{T}\int_{-T}^{T} f(y)\cos\left(\frac{n\pi}{T}y\right)dy \qquad (n \geq 1) \\ b_n = \frac{1}{T}\int_{-T}^{T} f(y)\sin\left(\frac{n\pi}{T}y\right)dy. \end{cases}$$

is called Fourier series of the 2T-periodic function f.

§1.2 Riemann-Lebesgue Lemma

Recall: A step function on $[-\pi, \pi]$ is a function of the form

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

where (i) $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

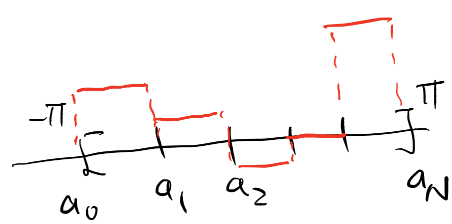
$I_0 = [a_0, a_1]$ with

$$-\pi = a_0 < a_1 < \cdots < a_{N-1} < a_N = \pi.$$

(ii) For a set E, $\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

is the characteristic function for E.

(iii) $s_j \in \mathbb{R}$, $j = 0, \cdots, N-1$



Lemma (Lemma 1.3 in KSChou's note)

Let f be an integrable on $[-\pi, \pi]$. Then $\forall \varepsilon > 0$, $\exists$ a step function $s(x)$ such that

(i) $s \leq f$ on $[-\pi, \pi]$, &

(ii) $\int_{-\pi}^{\pi} (f - s) < \varepsilon$.

Pf: f (Riemann) integrable

$\Rightarrow$ f can be approximated from below by <u>Darbour lower sums</u>

i.e. $\forall \varepsilon > 0$, $\exists$ partition $a_0 = -\pi < a_1 \cdots < a_N = \pi$

s.t.
$$\int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) < \varepsilon$$

where $m_j = \inf \{ f(x) : x \in [a_j, a_{j+1}] \}$.

Define the step function
$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j} \qquad (\text{ie. } s_j = m_j)$$

with $I_j = (a_j, a_{j+1}]$ for $j = 1, \ldots, N-1$
$I_0 = [a_0, a_1]$

Then $s \le f$ & $\int_{-\pi}^{\pi} s(x)\,dx = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j)$

$$\Rightarrow \int_{-\pi}^{\pi} (f - s) < \varepsilon. \quad \#$$

<u>Lemma 1.2</u> For every step function s integrable on $[-\pi, \pi]$, $\exists$ constant $C > 0$ (indep. of n, but depends on s) such that
$$|a_n(s)|,\ |b_n(s)| \le \frac{C}{n}, \quad \forall n \ge 1$$
where $a_n(s), b_n(s)$ are Fourier coefficients of s.

Pf: Let $s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}(x)$

We have, for $n \geq 1$

$$\pi a_n(s) = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{N-1} s_j \chi_{I_j}(x) \right) \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \frac{[\sin(n a_{j+1}) - \sin(n a_j)]}{n}$$

$$\Rightarrow \quad |a_n(s)| \leq \frac{C}{n}$$

where $C = \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j| > 0$ (indep on n)

Similarly for $|b_n(s)| \leq \frac{C}{n}$ for all $n \geq 1$. #

Now we can prove

Thm 1.1 (Riemann-Lebesgue Lemma)
The Fourier coefficients of a 2π-periodic function f integrable on $[-\pi, \pi]$ converge to 0 as $n \to +\infty$.

Pf: $\forall \varepsilon > 0$, Lemma 1.3 $\Rightarrow$ $\exists$ step function s s.t.

$$s \leq f \quad \& \quad \int_{-\pi}^{\pi} (f - s) < \frac{\varepsilon}{2}.$$

& by Lemma 1.2, $\exists n_0 > 0$ s.t.

$$|a_n(s)| < \frac{\varepsilon}{2} \quad \forall n \geq n_0$$

(for instance, take $n_0 = \left[\frac{2C}{\varepsilon}\right] + 1$, where C is the constant given by Lemma 1.2)

Therefore
$$|a_n(f) - a_n(s)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi}(f-s)(x)\cos nx\,dx\right|$$
$$\leq \frac{1}{\pi}\int_{-\pi}^{\pi} f-s \quad (\text{as } f \geq s)$$
$$< \frac{\varepsilon}{2\pi}$$

Hence
$$|a_n(f)| \leq |a_n(s)| + |a_n(f) - a_n(s)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\pi} < \varepsilon, \quad \forall n \geq n_0$$

$\therefore \quad a_n(f) \to 0$ as $n \to +\infty$.

Similarly for $b_n(f)$. ※

§1.3 Convergence of Fourier Series

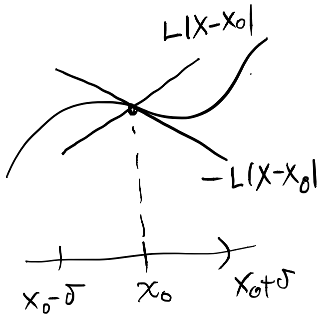
Terminology: For $f \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

we denote $(S_n f)(x) = a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$

the n-th partial sum of the Fourier series of f.

<u>Def</u>: Let f be a function on $[a,b]$. Then f is called <u>Lipschitz continuous</u> at $x_0 \in [a,b]$ if $\exists\, L>0$ & $\delta>0$ such that

$$|f(x)-f(x_0)| \le L\,|x-x_0|, \quad \forall\, |x-x_0|<\delta \quad (x\in[a,b])$$



<u>Note</u> (1) Both L & δ may depend on x_0.

<u>Note</u>: (2) If f is Lipschitz continuous at $x_0\in[a,b]$ & f is <u>bounded</u> on $[a,b]$, then $\exists\, L'>0$ (L' may depends on x_0) s.t.

$$|f(x)-f(x_0)| \le L'\,|x-x_0|.$$

<u>Pf</u>: By defn. f Lip. cts at x_0

$\Rightarrow \exists\, L>0,\ \delta>0$ s.t.

$$|f(x)-f(x_0)| \le L\,|x-x_0|, \quad \forall\, |x-x_0|<\delta.$$

If $|x-x_0|\ge\delta$, then $\dfrac{|x-x_0|}{\delta}\ge 1$

$$\Rightarrow |f(x)-f(x_0)| \le |f(x)|+|f(x_0)| \le 2M \le \frac{2M}{\delta}\,|x-x_0|$$

where $M=\sup\limits_{[a,b]}|f| \ge 0$.

Hence

$$|f(x)-f(x_0)| \leq \begin{cases} L|x-x_0| , & |x-x_0|<\delta \\ \frac{2M}{\delta}|x-x_0| , & |x-x_0| \geq \delta \end{cases}$$

$$\Rightarrow \quad |f(x)-f(x_0)| \leq L'|x-x_0| , \quad \forall x \in [a,b],$$

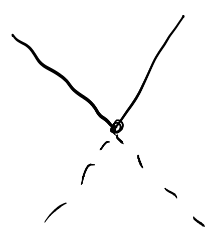
with $L' = \max\{L, \frac{2M}{\delta}\} > 0$. $\#$

eg: $f \in C^1[a,b]$ (continuously differentiable on $[a,b]$)

$\Rightarrow$ f is Lipschitz cts. at every $x_0 \in [a,b]$.

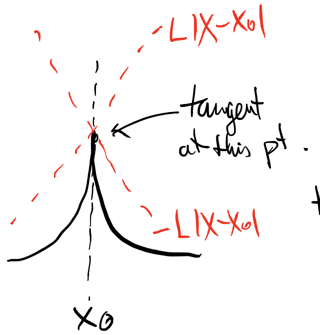
But $f(x)=|x|$ is Lip cts. at $x=0$, but not differentiable.

(Ex!)



(take $L=1$)

eg:



this graph gives a cts function at x_0, but not Lip. cts at x_0.

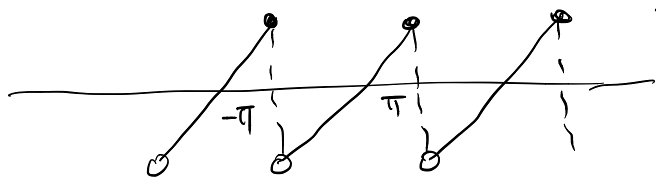
more precisely $f(x) = |x|^{\alpha}$ with $0<\alpha<1$ is not Lip cts. at $x=0$.

Thm 1.5 Let f be a 2π-periodic function integrable on $[-\pi,\pi]$. Suppose that f is Lipschitz continuous at x. Then $\{S_n f(x)\}$ converges to $f(x)$ as $n \to +\infty$.

(Pf: later at the end of this section)

eg of application:

Recall $f_1(x) = x$ on $[-\pi, \pi]$



$\hat{f}_1$ 2π-periodic extension

Fourier series $\quad x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$

It is clear that $f_1(x)$ is Lip. cts at any $x \in (-\pi, \pi)$

$$\therefore \quad \lim_{N\to+\infty} 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin(nx) = x \quad \forall\, x \in (-\pi, \pi).$$

On the other hand, $\hat{f}_1$ is discts. at $x = \pm\pi$,

and we've have seen $\quad \hat{f}_1(\pm\pi) \neq 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pm\pi).$

Thm 1.6 Let f be a 2π-periodic function integrable on $[-\pi, \pi]$.
Suppose that for $x_0 \in [-\pi, \pi]$

(i) $f(x_0^+) = \lim_{x\to x_0^+} f(x)$, $\quad f(x_0^-) = \lim_{x\to x_0^-} f(x)$ both exist.
(right-hand limit) $\quad$ (left-hand limit)

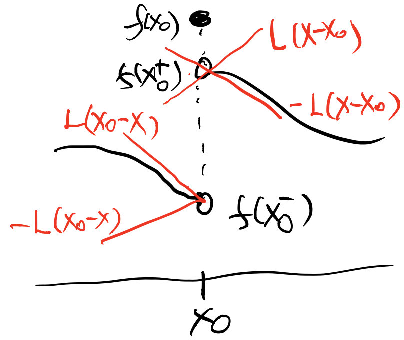
(ii) $\exists\, L > 0$ and $\delta > 0$ such that

$$|f(x) - f(x_0^+)| \le L(x - x_0), \quad 0 < x - x_0 < \delta$$

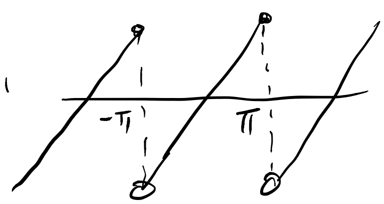
$$\& \quad |f(x) - f(x_0^-)| \le L(x_0 - x), \quad 0 < x_0 - x < \delta.$$

Then $\quad S_n f(x_0) \longrightarrow \dfrac{f(x_0^+) + f(x_0^-)}{2}$ as $n \to +\infty$.

(Pf = Omitted)



eg of application : $\quad f_1(x) = x$ with $\tilde{f}_1$



At $x_0 = \pi$, $\tilde{f}_1$ is discontinuous

(i) $\quad f(\pi^+) = \lim\limits_{x \to \pi^+} \tilde{f}_1(x) = -\pi$

$\quad f(\pi^-) = \lim\limits_{x \to \pi^-} \tilde{f}_1(x) = \pi$

(ii) $\quad$ For $0 < x - x_0 < \frac{\pi}{2}$ $\left(\text{ie } 0 < x - \pi < \frac{\pi}{2} \right)$

$(\delta = \frac{\pi}{2})$

we have

$$
\begin{aligned}
& |f(x) - f(\pi^+)| \\
&= |f(x - 2\pi) - (-\pi)| \qquad (x - 2\pi \in (-\pi, \pi)) \\
&= |x - 2\pi - (-\pi)| \\
&= x - \pi \le L(x - \pi) \text{ with } L = 1.
\end{aligned}
$$

Similar for $0 < x_0 - x < \frac{\pi}{2}$.

Hence conditions of Thm 1.6 are satisfied

$\Rightarrow$ Fourier series $\underset{\overset{\|}{0}}{S_n f(\pi)} \to \dfrac{f(\pi^+)+f(\pi^-)}{2} = \dfrac{-\pi+\pi}{2}$

$= 0$.

Next we turn to "uniform" convergence and need

<u>Def</u>: A function f defined on $[a,b]$ is called to <u>satisfy</u> <u>a</u> <u>Lipschitz</u> <u>condition</u> if $\exists L>0$ such that

$$|f(x)-f(y)| \le L|x-y|, \quad \forall x,y \in [a,b].$$

<u>Notes</u>: (1) $L>0$ is indep. of $x,y \in [a,b]$

a kind of "uniform" Lip condition.

(2) f satisfies a <u>Lip. condition</u> $\Rightarrow$ f is Lip cts at every point on $[a,b]$.

<u>eg</u>: If $f \in C^1[a,b]$, $\Rightarrow$ $|f(y)-f(x)| = \left|\int_x^y f'(t)\,dt\right|$

$\le M|y-x|, \quad \forall x,y \in [a,b]$

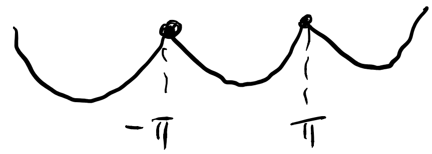
where $M = \sup_{[a,b]} |f'|$.

But $f(x)=|x|$ satisfies a Lip. condition, but not C^1.

<u>Thm1.7</u> Let f be a 2π-periodic function satisfying a Lipschitz condition. Then its Fourier series converges <u>uniformly</u> to f itself. (Pf: Omitted)

<u>eg of application</u> $f_2(x) = x^2$ on $[-\pi, \pi]$

$\tilde{f}_2$ 2π-periodic extension =



$\tilde{f}_2$ satisfies a Lip. condition (Check!)

$\Rightarrow$ $$\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$ converges uniformly

to x^2 on $[-\pi, \pi]$. ※

(Ex: put $x=0$ and get $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.)