Ch 1 Fourier Series

Def : (1) Trigonometric series (三角級數) on $[-\pi, \pi]$ is a series of functions of the form

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(where $a_n, b_n \in \mathbb{R}$)

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(2) If $b_n = 0$, $\forall n$, it is called a cosine series

" $a_n = 0$, $\forall n \geq 0$ " " . . . sine series.

Easy facts

(1) If $\sum_{n=0}^{\infty} |a_n|$ $\sum_{n=0}^{\infty} |b_n| < \infty$,

then $\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$

is uniformly and absolutely convergent.

In particular, $|a_n|, |b_n| \leq \frac{C}{n^s}$, $s > 1$ (for some $C > 0$), this is the case! (Pf: By M-test & $|\cos nx| \leq 1$, $|\sin nx| \leq 1$)

(2) $\phi(x) \overset{\text{def}}{=\!=} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

is a <u>continuous</u> <u>function</u> on $[-\pi, \pi]$ provided $\sum |a_n| < \infty$, $\sum |b_n| < \infty$.

(3) $\phi(x)$ defined in (2) is <u>2π-periodic</u>.

<u>Pf</u> $\phi(x+2\pi) = \lim_{n\to\infty} \sum_{k=0}^{n} (a_k \cos(k(x+2\pi)) + b_k \sin(k(x+2\pi)))$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)$$

$$= \phi(x)$$

※

<u>Def</u>: Let f be a 2π-periodic function on $\mathbb{R}$ which is <u>Riemann integrable</u> on $[-\pi, \pi]$. Then the <u>Fourier Series</u> (or <u>Fourier expansion</u>) of f is the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$\boxed{\text{Fourier coefficients of } f}$

$$\begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\,dy \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny\,dy \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny\,dy \end{cases}$$

<u>Notes</u> (1) $a_0 =$ average of f over $[-\pi,\pi]$

(2) Fourier series depends on the <u>global</u> <u>information</u> of f on $[-\pi,\pi]$.

(3) $f_1 \equiv f_2$ almost everywhere on $[-\pi,\pi]$

$\Rightarrow$ f_1 & f_2 have the <u>same</u> Fourier Series.

(4) Fourier series of f depends only on $f|_{(-\pi,\pi)}$, independent of the values of f on the end points.

<u>Motivation of the definition of Fourier Series</u>:

"If" $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$, $\forall x \in \mathbb{R}$

(& assume uniformly convergent.)

Then

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right)$$

It is easy to calculate

- $$\int_{-\pi}^{\pi}\cos mx\,dx = \begin{cases} 2\pi & \text{if } m=0 \\ 0 & \text{if } m\neq 0 \end{cases}$$

- $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases}$$

- $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0, \quad \forall n, m \geq 1$$

Hence if $m=0$,

$$\left.\begin{aligned} \text{L.H.S.} &= \int_{-\pi}^{\pi} f(x)\,dx \\ \text{R.H.S.} &= 2\pi a_0 \end{aligned}\right\} \Rightarrow \underline{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

if $m>0$, then

$$\left.\begin{aligned} \text{L.H.S.} &= \int_{-\pi}^{\pi} f(x)\cos mx\,dx \\ \text{R.H.S.} &= a_m \cdot \pi \end{aligned}\right\} \Rightarrow \underline{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx}$$

Similarly,

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left(a_n \overbrace{\int_{-\pi}^{\pi}\cos nx \sin mx\,dx}^{0} + b_n\int_{-\pi}^{\pi}\sin nx \sin mx\,dx \right)$$

Using $\int_{-\pi}^{\pi} \sin mx\,dx = 0, \ \forall m$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

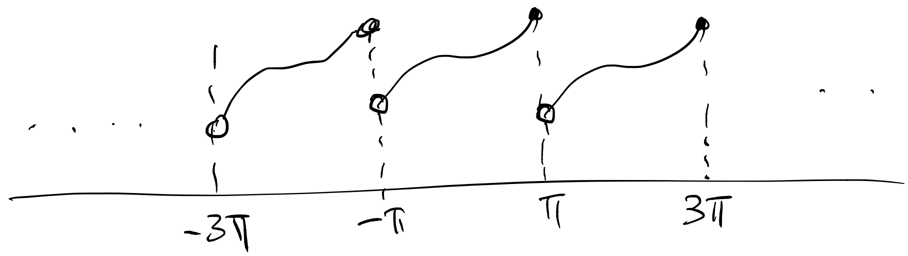
Hence

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

__Note__: For any Riemann integrable function f on $[-\pi, \pi]$, we can define all the a_0, a_n, b_n, $n \geq 1$ as in the defn.

& hence a Fourier series.

On the other hand, we can restrict a f to $(-\pi,\pi]$ and extend periodically to a 2π-periodic function $\tilde{f}$ on $\mathbb{R}$:



And according to the defn. of Fourier coefficients, f & $\tilde{f}$ have the same Fourier series!

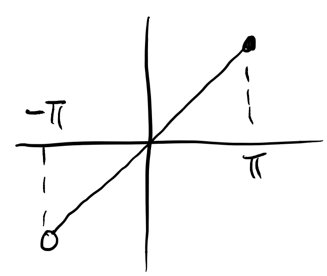
So we will not distinguish f & $\tilde{f}$!

Notation We use

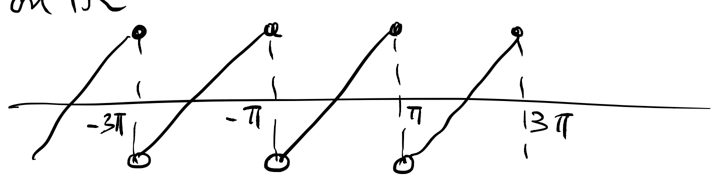
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

to denote if the trigonometric series on the RHS is the Fourier series of f.

eg 1.1 $f_1(x) = x$ restricted to $(-\pi,\pi]$



Extension to 2π-periodic function $\tilde{f}_1$ on $\mathbb{R}$



$\tilde{f}_1(-\pi) = \pi$
$\neq -\pi$

$\tilde{f}_1 =$ odd function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx = (-1)^{n+1}\frac{2}{n} \quad \text{(Check!)}$$

$$\therefore \quad f_1(x) = x \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx$$

$(\sim \tilde{f}_1(x))$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx \quad \text{is a sine series}$$

($\because f_1$ is odd)

Notes : (1) For $x = \pm\pi$, Fourier series $\Big|_{\pm\pi} = 0$

But $\left.\begin{array}{l} f_1(\pm\pi) = \pm\pi \\ \tilde{f}_1(\pm\pi) = \pi \end{array}\right\} \neq$ Fourier series $\Big|_{\pm\pi}$

(2) Convergence is not clear (for $x \neq \pm\pi$)

as the terms decay like $\frac{1}{n}$ & $\sum \frac{1}{n}$ doesn't converge.

Notation : "Big O" & "small o"

Let $\{x_n\}$ be a sequence, then

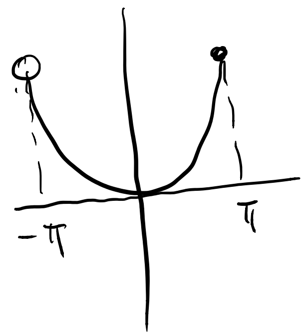
(i) $x_n = O(n^s) \Leftrightarrow |x_n| \leq C n^s$ for some const. $C > 0$ (as $n \to \infty$)

| (ii) $x_n = o(n^s) \Leftrightarrow \dfrac{|x_n|}{n^s} \to 0$ as $n \to \infty$

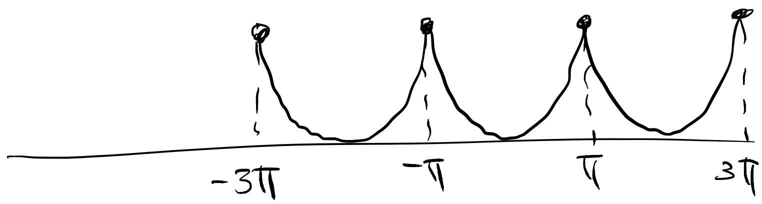
(egs): (i) $x_n = \dfrac{2(-1)^{n+1}}{n} \sin nx = O(\frac{1}{n})$ $\left(|x_n| \le \frac{2}{n} \right)$

(ii) $x_n = \log n = o(n)$ $\left(\frac{\log n}{n} \to 0 \text{ as } n \to \infty \right)$

<u>Eg 1.2</u> $f_2(x) = x^2$ restricted to $(-\pi, \pi]$



Extension to a 2π-periodic function $\tilde{f}_2$ on $\mathbb{R}$



$\tilde{f}_2$ is continuous (since $f_2(-\pi) = f_2(\pi)$)

$\tilde{f}_2$ is an even function

It is an <u>easy exercise</u> of integration to find that

$$f_2(x) \equiv x^2 \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \quad (\text{Ex!})$$

One sees that $a_n = O(\frac{1}{n^2}) \Rightarrow \sum |a_n| < \infty$

$\Rightarrow$ Fourier series converges uniformly to a continuous function.

(Will it be the function $\tilde{f}_2$? See later discussion)

<u>Observation</u>: Egs 1 & 2: $\begin{cases} \text{odd function} \longrightarrow \text{Sine series} \\ \text{even function} \longrightarrow \text{Cosine series} \end{cases}$

This is true in general ! (Ex !)