Notes : (1) z_0 = removable singular point of f

$$\Rightarrow \quad f(z) = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

$$0 < |z-z_0| < R_1$$

and can be extended to an analytic function :

$$f(z) = \begin{cases} f(z) & , \quad 0<|z-z_0|<R_1 \\ a_0 & , \quad \text{if } z = z_0 \end{cases}$$

on $\{ |z-z_0| < R_1 \}$.

(2) z_0 = pole of order m of f.

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m},$$

$$b_m \neq 0 \;, \quad 0<|z-z_0|<R_2 .$$

$$\Rightarrow \quad (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_m$$

has a removable singular point at $z=z_0$ & can be extended to an analytic function in $\{ |z-z_0| < R_2 \}$.

eg1 : $f(z) = \dfrac{1-\cosh z}{z^2}$ is analytic in $0<|z|<\infty$

Note that $f(z) = \dfrac{1}{z^2}\left(1-\cosh z\right)$

$$= \frac{1}{z^2}\left[1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\right]$$

$$= \frac{1}{z^2}\left[-\frac{z^2}{2!}-\frac{z^4}{4!}-\cdots\right]$$

$$= -\frac{1}{2!}-\frac{z^2}{4!}-\cdots$$

$\Rightarrow$ $z=0$ is a removable singular point of $f(z)$.

$$f(z)=\begin{cases} \dfrac{1-\cosh z}{z^2}, & 0<|z|<\infty \\ -\dfrac{1}{2}, & z=0 \end{cases}$$

is analytic in $\{|z|<\infty\}$ (i.e. an entire function)

eg2: $e^{1/z} = 1+\dfrac{\left(\frac{1}{z}\right)}{1!}+\dfrac{\left(\frac{1}{z}\right)^2}{2!}+\cdots$

$$= 1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\cdots+\frac{1}{n!}\frac{1}{z^n}+\cdots$$

$(0<|z|<\infty)$

infinitely many nonzero terms in the principal part of the Laurent series,

$\therefore$ $z=0$ is an essential singular point of $e^{1/z}$.

eg3: $f(z)=\dfrac{1}{z^2(1-z)}$

$$= \frac{1}{z^2}\left(1+z+z^2+\cdots\right) \qquad 0<|z|<1$$

$$= \frac{1}{z^2}+\frac{1}{z}+1+z+\cdots \qquad 0<|z|<1$$

$\Rightarrow$ $z=0$ is a pole of order 2 of $f(z)$.

$\Rightarrow$ $z^2 f(z) = \frac{1}{1-z}$ analytic in $\{|z|<1\}$

(not at $z=1$)

eg 4 : $f(z) = \dfrac{z^2+z-2}{z+1}$

$$= -\frac{2}{z+1} - 1 + (z+1)$$

$\Rightarrow$ $z=-1$ is a simple pole of $f(z)$.

§80 Residues at Poles

Thm Let z_0 be an isolated singular point of a function f. The following are equivalent :

(a) z_0 is a pole of order m ($m=1,2,\cdots$) of f.

(b) $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad (m=1,2,\cdots)$$

where $\phi(z)$ is analytic and nonzero at z_0.

Moreover, if (a) & (b) are true, then

$$\operatorname{Res}_{z=z_0} f(z) = \begin{cases} \phi(z_0), & \text{if } m=1 \\ \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}, & \text{otherwise} \end{cases}$$

Pf: (a) $\Rightarrow$ (b)

By Note 2:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad b_m \neq 0$$

$$\Rightarrow (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_m$$

Then

$$\phi(z) = \begin{cases} b_m + \cdots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}, & 0<|z-z_0|<R \\ b_m, & z=z_0 \end{cases}$$

is analytic in $\{|z-z_0|<R\}$ & $\phi(z_0) = b_m \neq 0$.

$$\Rightarrow \quad f(z) = \frac{\phi(z)}{(z-z_0)^m}.$$

(b) $\Rightarrow$ (a) If $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$, $\phi(z)$ analytic in $\{|z-z_0|<R\}$

& $\phi(z_0) \neq 0$. Then

$$\phi(z) = a_0 + a_1 (z-z_0) + \cdots + a_n (z-z_0)^n + \cdots$$
$$(|z-z_0|<R)$$

with $a_0 = \phi(z_0) \neq 0$

$$\Rightarrow \quad f(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots, \quad a_0 \neq 0$$

$\therefore \quad z = z_0$ is a pole of order m of f.

Moreover, from

$$f(z) = \frac{a_0}{(z-z_0)^m} + \cdots + \frac{a_{m-1}}{z-z_0} + \cdots$$

$$\therefore \quad \operatorname{Res}_{z=z_0} f(z) = a_{m-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \#$$

$\left(\text{For } m=1, \ \phi^{(0)}(z_0) = \phi(z_0), \ 0! = 1 \right)$

§81 Examples

eg1 : $f(z) = \dfrac{z+4}{z^2+1}$ has isolated singular point at $z = i$

$$= \frac{z+4}{(z+i)(z-i)} = \frac{\left(\frac{z+4}{z+i}\right)}{z-i} = \frac{\phi(z)}{z-i}$$

where $\phi(z) = \dfrac{z+4}{z+i}$ analytic at $z = i$.

and $\phi(i) = \dfrac{i+4}{2i} = \dfrac{1}{2} - 2i \neq 0$

$\therefore$ $z = i$ is a simple pole of f and

$$\operatorname{Res}_{z=i} \frac{z+4}{z^2+1} = \frac{1}{2} - 2i.$$

eg2 : If $f(z) = \dfrac{z^3+2z}{(z-i)^3}$.

Since $\phi(z) = z^3 + 2z$ is analytic at $z = i$

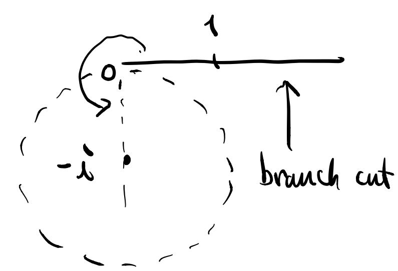
and $\phi(i) = i^3 + 2i = i \neq 0$

$\therefore$ $z = i$ is a pole of order 3 of $f(z) = \dfrac{z^3+2z}{(z-i)^3}$

$$\therefore \quad \operatorname{Res}_{z=i} f(z) = \frac{\phi^{(2)}(i)}{2!} = \frac{(6z)\big|_{z=i}}{2} = 3i$$

eg 3: Let $f(z) = \dfrac{(\log z)^3}{z^2+1}$ where the branch of log is $\log z = \ln r + i\theta$, $0 < \theta < 2\pi$.

Since this branch of log is analytic in $0 < |z+i| < 1$, we have



$$f(z) = \frac{(\log z)^3}{(z-i)(z+i)} = \frac{\left(\dfrac{(\log z)^3}{z-i}\right)}{z+i} = \frac{\phi(z)}{z+i}$$

where $\phi(z) = \dfrac{(\log z)^3}{z-i}$ analytic in $\{|z+i| < 1\}$

and $\phi(-i) = \dfrac{(\log(-i))^3}{-2i} = \dfrac{\left(i\frac{3\pi}{2}\right)^3}{-2i} = +\dfrac{27\pi^3}{16} \neq 0$

$\therefore$ $z = -i$ is a simple pole of $f(z)$

with $\operatorname{Res}_{z=-i} \dfrac{(\log z)^3}{z^2+1} = \dfrac{27\pi^3}{16}$.

eg 4: $f(z) = \dfrac{1-\cos z}{z^3}$, Note that $(1-\cos z)\big|_{z=0} = 0$

$\therefore$ $z=0$ is <u>not</u> a pole of order 3.

Instead

$$f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\right]$$

$$= \frac{1}{z^3}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots\right]$$

$$= \frac{1}{2}\cdot\frac{1}{z} - \frac{z}{4!} + \cdots$$

$\therefore$ $z=0$ is a simple pole of $\frac{1-\cos z}{z^3}$.

$$\Rightarrow f(z) = \frac{\left(\frac{1-\cos z}{z^2}\right)}{z} = \frac{\phi(z)}{z}$$

where $\phi(z) = \frac{1-\cos z}{z^2}$ analytic & nonzero at $z=0$

$$\left(\phi(0) = \frac{1}{2} \neq 0\right)$$

$$\Rightarrow \operatorname{Res}_{z=0} \frac{1-\cos z}{z^3} = \frac{1}{2}. \left(= \lim_{z\to 0} \frac{1-\cos z}{z^2}\right)$$

<u>eg5</u>: $f(z) = \frac{1}{z^2 \sinh z} = \frac{\left(\frac{z}{\sinh z}\right)}{z^3} = \frac{\phi(z)}{z^3}$

where $\phi(z) = \frac{z}{\sinh z} = \frac{z}{z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots} = \frac{1}{1+\frac{z^2}{3!}+\frac{z^5}{5!}+\cdots}$

$$= 1 - \frac{z^2}{3!} + \cdots \qquad \text{for } 0<|z|<R$$
(some small R)

with $\phi(0)=1$.

$\therefore$ $z=0$ is a pole of order 3 of $f(z)$

& $\displaystyle \operatorname{Res}_{z=0} \frac{1}{z^2 \sinh z} = \frac{1}{2!} \frac{d^2}{dz^2}\Big|_{z=0} \left(\frac{z}{\sinh z} \right)$

Of course, it is easier to find the residue using series expansion than calculating $\phi^{(2)}(z_0)$. (in this case.)

§82 Zeros of Analytic Functions

Def: Suppose f is analytic at z_0. If there is a positive integer $m \geq 1$ such that

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \text{and}$$

$$f^{(m)}(z_0) \neq 0,$$

then f is said to have a zero of order m at z_0.

Thm 1: Let f be analytic at z_0. Then f has a zero of order m at z_0 if and only if there is an analytic function $g(z)$ such that

$$f(z) = (z-z_0)^m g(z) \quad \text{and} \quad g(z_0) \neq 0.$$

Pf: $(\Rightarrow)$ Taylor's expansion

$$f(z) = \cancel{f^{(0)}}(z_0) + \frac{\cancel{f^{(1)}(z_0)}}{1!}(z-z_0) + \cdots$$

$$+ \frac{\cancel{f^{(m-1)}(z_0)}}{(m-1)!}(z-z_0)^{m-1} + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m \qquad f^{(m)} \neq 0$$

$$+ \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

z_0 is a zero of order m $\Rightarrow$

$$f(z) = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

$$= (z-z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots \right]$$

$$= (z-z_0)^m g(z)$$

where $g(z) = \dfrac{f^{(m)}(z_0)}{m!} + \dfrac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots$

is analytic at z_0 with

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0 .$$

($\Leftarrow$) If g analytic and nonzero at z_0,

then $g(z) = g(z_0) + g'(z_0)(z-z_0) + \cdots$

$\Rightarrow f(z) = (z-z_0)^m g(z)$

$= g(z_0)(z-z_0)^m + g'(z_0)(z-z_0)^{m+1} + \cdots$

By uniqueness of power series expansion,

$f^{(0)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and

$f^{(m)}(z_0) = m!\, g(z_0) \neq 0 .$

$\therefore$ z_0 is a zero of order m. #

<u>Thm 2</u> Suppose f is analytic and z_0 is a zero of f, but $f(z) \not\equiv 0$ in any neighborhood of z_0. Then $\exists\, \varepsilon > 0$ such that $f(z) \neq 0$, $\forall z \in \{0 < |z-z_0| < \varepsilon\}$.

(i.e. only z_0 is a zero of f in $\{|z-z_0| < \varepsilon\}$, in other words, zeroes of f are isolated.)

Pf: $f \not\equiv 0$ in any neighborhood of z_0

$\Rightarrow$ Taylor's expansion of f about $z_0 \neq 0$.

$\Rightarrow$ z_0 is a zero of order m for some finite $m \geq 1$.

$\Rightarrow$ $f(z) = (z-z_0)^m g(z)$ (by Thm 1)

with $g(z)$ analytic & nonzero at z_0.

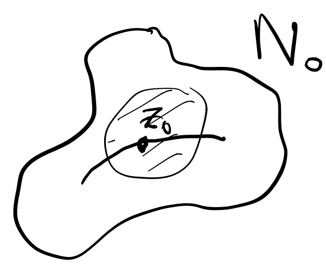
Continuity of $g(z)$ $\Rightarrow$ $\exists\, \varepsilon > 0$ such that

$g(z) \neq 0 \quad \forall z \in \{ |z-z_0| < \varepsilon \}$.

$\therefore \quad f(z) = (z-z_0)^m g(z) \neq 0, \ \forall z \in \{ 0 < |z-z_0| < \varepsilon \}$.

#

Thm 3: Suppose that f is analytic in a neighborhood N_0 of z_0 and $f(z)=0$ at each point z of a domain or line segment containing z_0. Then $f(z) \equiv 0$ in N_0.



Pf: By assumption, $f(z)=0$ for some points in $0 < |z-z_0| < \varepsilon$, for any $\varepsilon > 0$.

$\therefore$ Thm 2 $\Rightarrow$ $f \equiv 0$ in the nbd of z_0 $\Rightarrow$ $f \equiv 0$ in N_0 #