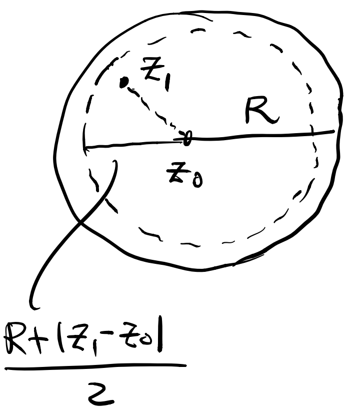
§70 Continuity of Sums of Power Series

Thm: A power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0|=R$.



Pf: Let $z_1 \in \{|z-z_0|<R\}$

And set $R_0 = \frac{R+|z_1-z_0|}{2}$

Then

$$|z_1 - z_0| < R_0 < R$$

By Thm 2 in §69, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges uniformly on $|z-z_0| \le R_0$. i.e.

$\forall \varepsilon > 0$, $\exists N_\varepsilon$ (indep. of z in $|z-z_0| \le R_0$) such that

$$|\rho_N(z)| < \frac{\varepsilon}{3}, \quad \forall N > N_\varepsilon, \; z \in \{|z-z_0| \le R\}.$$

Since $(N_\varepsilon+1)$th partial sum $S_{N_\varepsilon+1}(z)$ is a polynomial, it is continuous (in $\mathbb{C}$). Therefore, $\exists \delta > 0$ such that

$$|S_{N_\varepsilon+1}(z) - S_{N_\varepsilon+1}(z_1)| < \frac{\varepsilon}{3}, \quad \forall |z-z_1| < \delta.$$

All together, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall\ |z - z_1| < \delta$, we have

$$|S(z) - S(z_1)| \le |S(z) - S_{N_\varepsilon+1}(z)| + |S_{N_\varepsilon+1}(z) - S_{N_\varepsilon+1}(z_1)| + |S_{N_\varepsilon+1}(z_1) - S(z_1)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\therefore$ $S(z)$ is cts at z_1.

Since $z_1 \in \{|z - z_0| < R\}$ is arbitrary, $S(z)$ is cts on $\{|z - z_0| < R\}$. ※

<u>Note</u>: One can modify the proofs of Thms 1 & 2 in §69 to conclude that if a Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ is valid in

$R_1 < |z - z_0| < R_2$, then <u>both</u> series in the expansion converge absolutely and uniformly in any $r_1 \le |z - z_0| \le r_2$ with $R_1 < r_1 < r_2 < R_2$ ※

§71 Integration and Differentiation of Power Series

Thm 7 Let C be a contour interior to the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, and $g(z)$ be any function that is continuous on C. Then the series (of cpx numbers) $\sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz$

converges and

$$\boxed{\int_C g(z) \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz}$$

(i.e. $\sum_{n=0}^{\infty} a_n g(z)(z-z_0)^n$ can be integrated term-by-term.)

Pf: Let $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Then the Thm in §70 $\Rightarrow$ $S(z)$ is continuous on $\{|z-z_0|<R\}$ where $|z-z_0|=R$ is the circle of convergence.

$\Rightarrow$ Both $g(z)$ & $S(z)$ are cts. on C and the product can be written as

$$g(z) S(z) = \sum_{n=0}^{N-1} a_n g(z)(z-z_0)^n + g(z)\rho_N(z),$$

where $\rho_N(z) = \sum_{n=N}^{\infty} a_n(z-z_0)^n$ is the remainder.

$$\Rightarrow \int_C g(z) S(z) ds = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n dz + \int_C g(z)\rho_N(z) dz.$$

To estimate $\int_C g(z)\rho_N(z)dz$, we let

$$M = \max_{z\in C} |g(z)| \quad \& \quad L = \text{length of } C.$$

And by the uniform convergence of power series inside the circle of convergence, we have

$\forall \varepsilon > 0, \exists N_\varepsilon$ such that $\forall z \in C$,

$$|\rho_N(z)| < \varepsilon, \quad \forall N > N_\varepsilon.$$

Therefore $\left| \int_C g(z)\rho_N(z) dz \right| \leq M\varepsilon \cdot L$.

Letting $\varepsilon \to 0$, we see that $\lim_{N\to\infty} \int_C g(z)\rho_N(z)dz = 0$.

Hence $\int_C g(z) S(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz$

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<u>Cor</u>: The sum $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is <u>analytic</u> at each point z interior to the circle of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Pf: In Thm 1, take $g(z) \equiv 1$ and note that

$$\int_C (z-z_0)^n dz = 0 \quad \forall\, n = 0, 1, 3, \cdots,$$

for all closed contour, we have

$$\int_C S(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz = 0$$

for all closed contour. Then Thm 2 of §57 (9th Ed) (Morera Thm) implies $S(z)$ is analytic inside the circle of convergence. #

eg1: Show that $f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$

is an entire function.

Pf: Since $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ valid $\forall\, |z| < \infty$.

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$\Rightarrow$ For $z \neq 0$,

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$0 < |z| < \infty$

Note that for $z = 0$

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots$$

$$=1-0+0-\cdots \quad \text{is convergent at } z=0.$$

$$=1=f(0)$$

$$\therefore\ f(z)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \quad \text{valid on } |z|<\infty$$

Then by the Cor., $f(z)$ is entire. #

<u>Thm2</u> The power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated term-by-term inside the circle of convergence. That is

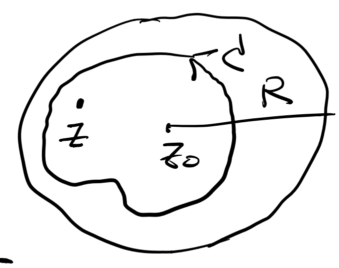
$$S'(z)=\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1},$$

$\forall z$ inside the circle of convergence, where

$$S(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

<u>Pf</u>: Let z be a point inside the circle of convergence.

Let C be a positively oriented simple closed contour surrounding z and interior to the circle of convergence.



Define $g(s)=\frac{1}{2\pi i}\frac{1}{(s-z)^2}$, $\forall s\in C$.

Then it is cts on C, and Thm 1 $\Rightarrow$

$$\int_C g(s) S'(s)\,ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s-z_0)^n ds$$

ie.

$$\frac{1}{2\pi i}\int_C \frac{S'(s)\,ds}{(s-z)^2} = \sum_{n=0}^{\infty} a_n \left(\frac{1}{2\pi i}\int_C \frac{(s-z_0)^n ds}{(s-z)^2}\right)$$

By Cauchy integral formula

$$S'(z) = \sum_{n=0}^{\infty} a_n \left[\frac{d}{ds}(s-z_0)^n\right]_{s=z}$$

$$= \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \quad \#$$

eg: For $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $(|z|<\infty)$.

$$\Rightarrow \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)z^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad (|z|<\infty) \quad \#$$

(eg 2 : Reading Ex!)