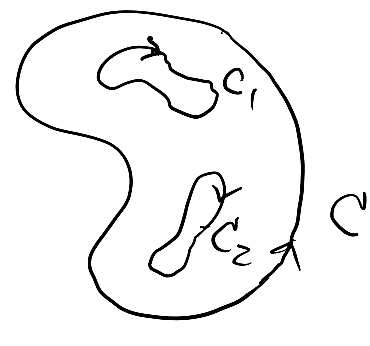
Thm: Suppose that

(a) C is a simple closed contour in counterclockwise direction

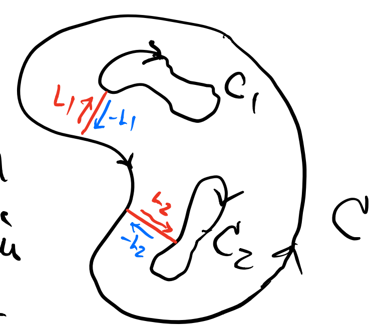
(b) C_k, $k=1,2,\cdots,n$ are simple closed contours interior to C in clockwise direction, they are disjoint and whose interiors are also disjoint.



If a function f is analytic on C and C_k, $k=1,2,\cdots,n$, and throughout the (multiply-connected) domain consisting of the points interior to C but exterior to C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0 .$$

Pf: Let L_i be polygonal path joining C to C_k, $k=1,\cdots,n$ in the multiply connected domain such that L_k has no self-



intersection and L_k are disjoint.

Then a simple closed contour Γ can be formed:

$$\Gamma = C + L_1 + C_1 + (-L_1) + \cdots + L_n + C_n + (-L_n).$$

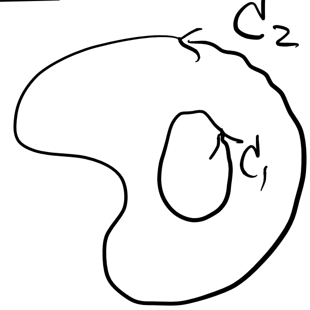
By Cauchy-Goursat Thm,

$$0 = \int_\Gamma f\,dz = \left(\int_C + \int_{L_1} + \int_{C_1} + \int_{(-L_1)} + \cdots + \int_{L_n} + \int_{C_n} + \int_{(-L_n)} \right) f\,dz$$

$$= \int_C f\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz. \quad \#$$

<u>Cor</u>: (<u>Principle of deformation of paths</u>)

Let C_1 & C_2 be <u>positively</u> oriented simply closed contours, where C_1 is interior to C_2.

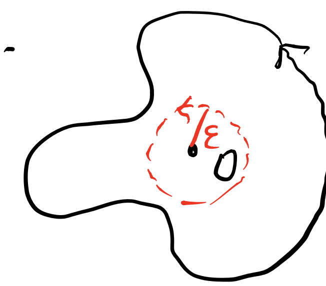


If f is analytic in the closed region consisting of C_1 & C_2 and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz .$$

Pf: By Thm, $\int_{C_2} f\,dz + \int_{-C_1} f(z)\,dz = 0$ ※

Eg: Let C = any positively oriented simply closed contour surrending the origin.



Then $\displaystyle\int_C \frac{dz}{z} = 2\pi i$

Pf: Choose $C_0 : z = \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$ with $\varepsilon > 0$ small enough s.t. $B_\varepsilon(0)$ is interior to C.

Then by corollary

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z} = \int_0^{2\pi} \frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = 2\pi i$$

as $f(z) = \frac{1}{z}$ is analytic on C & C_0 & between them.

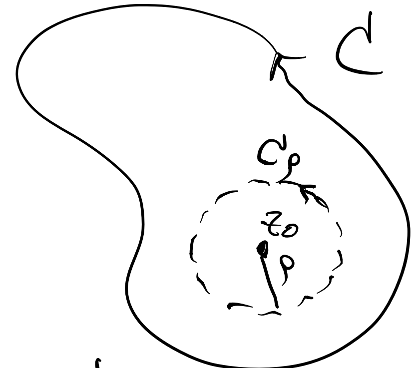
§54 Cauchy Integral Formula

<u>Thm</u>: Let f be analytic everywhere inside and on a simple closed contour C in positive orientation. If z_0 is any point interior to C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}}$$

(Cauchy Integral Formula)

<u>Pf</u>: Since z_0 is interior to C, $\forall\ \rho$ small enough, $B_\rho(z_0)$ is interior to C.



Let $C_\rho = \partial B_\rho(z_0) = \{\,|z-z_0| = \rho\,\}$ in positive orientation parametrized by

$$z = z_0 + \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Then by the Thm in previous section,

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

Since $\frac{f(z)}{z-z_0}$ is analytic on & between C' & C_ρ.

$\therefore$ $\forall \rho > 0$ small enough,

$$\int_{C'} \frac{f(z)\,dz}{z-z_0} = \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})\,d(z_0+\rho e^{i\theta})}{\rho e^{i\theta}}$$

$$= i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

As f analytic $\Rightarrow$ f cts at z_0

$\Rightarrow$ $\forall \varepsilon > 0, \exists \rho_0 > 0$ s.t.

$$|f(z_0+\rho e^{i\theta}) - f(z_0)| < \varepsilon, \quad \forall\, 0<\rho<\rho_0$$

$$\therefore \left| \int_{C'} \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right|$$

$$= \left| i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta - i\int_0^{2\pi} f(z_0)\,d\theta \right|$$

$$\leq \int_0^{2\pi} |f(z_0+\rho e^{i\theta}) - f(z_0)|\,d\theta$$

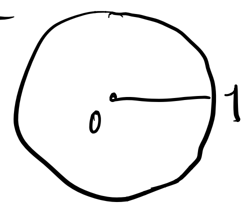
$$< 2\pi\varepsilon.$$

$$\Rightarrow \quad \int_{C'} \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0) \quad \#$$

eg: Let $f(z) = \dfrac{\cos z}{z^2+9}$, $C: z = e^{i\theta}$, $0 \le \theta \le 2\pi$

positive oriented unit circle

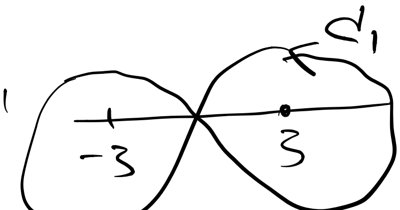
Then $f(z)$ is analytic on and inside C



$$\Rightarrow \int_C \frac{\cos z \, dz}{z(z^2+9)} = \int_C \frac{f(z)}{z} dz$$

$$= 2\pi i f(0) = \frac{2\pi i}{9} \quad \#$$

(try $\int_{C_1} f(z) dz$ where C_1



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§55 An Extension of the Cauchy Integral Formula

Notation: $f^{(n)}(z_0)$ denotes the n-th derivative of f at z_0, where $f^{(0)}(z_0) = f(z_0)$.

$$\left(f^{(n)}(z_0) = \frac{d}{dz} f^{(n-1)}(z_0) \right)$$

Thm: Let f be analytic inside and on a simple closed contour C taken in the positive sense. If z_0 is any point interior to C then $\forall\, n = 0, 1, 2, \cdots$,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$ (Cauchy Integral Formula)

Application:

eg1: If C = positively oriented unit circle.

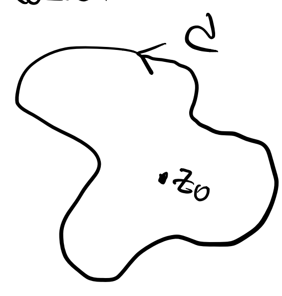
Then $$\int_C \frac{\exp(2z)}{z^4}\,dz = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} \quad \text{where } f(z)=e^{2z}$$

$$= \frac{2\pi i}{3!} f^{(3)}(0)$$

$$= \frac{8\pi i}{3}$$

eg2: C = positively oriented simply closed curve

z_0 interior to C.



Then applying the thm to

$f(z) \equiv 1$ with $n=0$

$$1 = \frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz$$

$$\therefore \int_C \frac{dz}{z-z_0} = 2\pi i$$

For $n = 1, 2, 3, \dots$

$$0 = f^{(n)}(z_0) = \int_C \frac{dz}{(z-z_0)^{n+1}} .$$

<u>Note</u>: Replace the dummy index of the integral by s and then let z_0 be a general point z interior to C. Then

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}}}$$

$\forall z$ interior to C, & $n = 0, 1, 2, \dots$

In particular

$$\boxed{f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}}$$

$\forall z$ interior to C.

<u>eg</u>: Let $f(z) = (z^2-1)^n$ $(n = 0, 1, 2, \dots)$ (entire function) Cauchy Integral Formula

$$\Rightarrow \quad \frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n\,ds}{(s-z)^{n+1}}$$

for simple closed contour C surrounding z.

The "Legendre Polynomial" $P_n(z)$ is defined as

$$P_n(z) = \frac{1}{n!\, 2^n} \frac{d^n}{dz^n} (z^2-1)^n \quad , \forall n = 0,1,2,\cdots$$

$$= \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n ds}{(s-z)^{n+1}} \quad , \forall n=0,1,2,\cdots$$

Let calculate $P_n(1)$. By Thm,

$$P_n(1) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2-1)^n ds}{(s-1)^{n+1}}$$

$$= \frac{1}{2^{n+1}\pi i} \int_C \frac{(s+1)^n}{s-1} ds$$

$$= \frac{1}{2^n} \left(\frac{1}{2\pi i} \int_C \frac{f(s) ds}{s-1} \right) \quad \text{where } f(s) = (s+1)^n$$

$$= \frac{1}{2^n} f(1)$$

$$= 1. \quad \#$$

<u>Note</u>: Since $\frac{\partial}{\partial z}\left(\frac{f(s)}{(s-z)^n} \right) = \frac{n f(s)}{(s-z)^{n+1}}$,

so "famally" $f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^{n+1}}$

$$= \frac{(n-1)!}{2\pi i} \int_C \frac{\partial}{\partial z}\left(\frac{f(s)}{(s-z)^n} \right) ds$$

$$\text{"="} \frac{d}{dz}\left(\frac{(n-1)!}{2\pi i} \int_C \frac{f(s)}{(s-z)^n} ds \right)$$

$$= \frac{d}{dz}\left(f^{(n-1)}(z)\right).$$

§56 Verification of the Extension

Proof of the case "$n=1$"

By Cauchy-Goursat $\quad f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}$,

C = simple closed contour surrending z.

$\Rightarrow$ for $|\Delta z|$ small enough (s.t. $z+\Delta z$ interia to C)

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{\Delta z}\cdot\frac{1}{2\pi i}\left(\int_C \frac{f(s)\,ds}{s-(z+\Delta z)} - \int_C \frac{f(s)\,ds}{s-z}\right)$$

$$= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-(z+\Delta z)} - \frac{1}{s-z}\right)\frac{f(s)\,ds}{\Delta z}$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{[s-(z+\Delta z)](s-z)} f(s)\,ds.$$

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}\right|$$

$$= \left|\frac{1}{2\pi i}\int_C \left[\frac{1}{[s-(z+\Delta z)](s-z)} - \frac{1}{(s-z)^2}\right] f(s)\,ds\right|$$

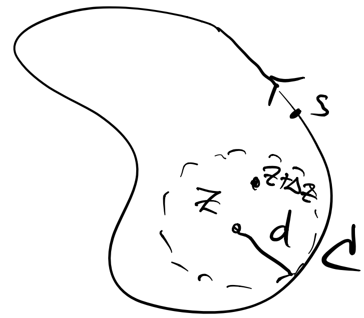
$$\le \frac{1}{2\pi}\int_C \left|\frac{\Delta z}{(s-(z+\Delta z))(s-z)^2}\right| |f(s)|\,ds$$

Let $d = \text{dist}(z, C)$

Then for $0 < |\Delta z| < d$,

$z + \Delta z$ is interior to C

and $\forall\, s \in C$,



$$d \le |z - s| \le |z - (z+\Delta z)| + |z + \Delta z - s|$$
$$= |\Delta z| + |s - (z+\Delta z)|$$

$\therefore \quad |s - (z+\Delta z)| \ge d - |\Delta z| > 0.$

$$\Rightarrow \quad \left| \frac{\Delta z}{(s-(z+\Delta z))(s-z)^2} \right| \le \frac{|\Delta z|}{(d - |\Delta z|)\, d^2}$$

$\therefore$

$$\left| \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)^2} \right| \le \frac{1}{2\pi} \frac{|\Delta z|}{(d - |\Delta z|) d^2} ML$$

where $M = \max_C |f(s)|$ & $L =$ length of C.

Since M, L, d are indep. of Δz, we have

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{(s-z)^2}$$

#

§57 Some Consequences of the Extension

Thm 1 : If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Pf: f analytic at z_0

$\xrightarrow{\text{by def}^n}$ f analytic in a nbd $\{|z-z_0|<\varepsilon\}$ of z_0.

$\Rightarrow$ f analytic inside and on the circle

$$C_0 : |z-z_0| = \frac{\varepsilon}{2}.$$

Then Cauchy Integral Formula

$$\Rightarrow \quad f^{(2)}(z) = \frac{2!}{2\pi i}\int_{C_0} \frac{f(s)\,ds}{(s-z)^3}, \quad \forall z \text{ interior to } C_0.$$

$\therefore$ $f^{(2)}(z)$ exists $\forall z \in B_{\varepsilon/2}(z_0)$.

$\Rightarrow$ $f^{(1)}(z)$ is analytic in $B_{\varepsilon/2}(z_0)$.

$\therefore$ f' is analytic at z_0.

Similar argument with mathematical induction

$\Rightarrow$ $f^{(n)}$ is analytic at z_0, $\forall n$. #

Cor: If a function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z=(x,y)$, then u and v have continuous partial derivatives of all order at that points.