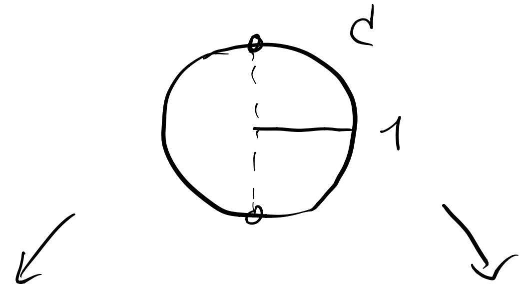
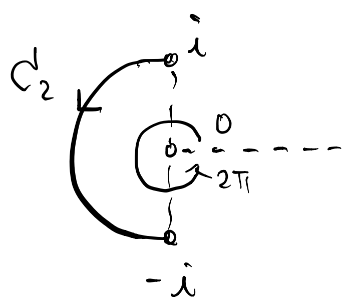
(cont'd from previous example)

So if we want to use antiderivative for $\int_C \frac{dz}{z}$, we need to divide the unit circle $C$ into 2 parts and use different branches of $\log z$ for different part.

For instance:




For this, one can use the branch of $\log z$ with

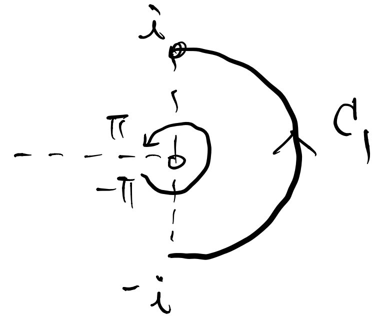
$$0 < \theta < 2\pi$$

$$\frac{d}{dz} \log z = \frac{1}{z}, \quad 0 < \theta < 2\pi$$

$$\Rightarrow \int_{C_2} \frac{dz}{z} = \log z \Big|_{i}^{-i} = \log z \Big|_{e^{i\frac{\pi}{2}}}^{e^{i\frac{3\pi}{2}}}$$

$$= \Big[\ln|z| + i\theta\Big]_{e^{i\frac{\pi}{2}}}^{e^{i\frac{3\pi}{2}}}$$

$$= i\frac{3\pi}{2} - i\frac{\pi}{2} = i\pi$$




For this, one can use the principal branch of $\log z$:

$$-\pi < \Theta < \pi$$

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}, \quad -\pi < \Theta < \pi$$

$$\Rightarrow \int_{C_1} \frac{dz}{z} = \operatorname{Log} z \Big|_{-i}^{i} = \operatorname{Log} z \Big|_{e^{-i\frac{\pi}{2}}}^{e^{i\frac{\pi}{2}}}$$

$$= \Big[\ln|z| + i\Theta\Big]_{e^{-i\frac{\pi}{2}}}^{e^{i\frac{\pi}{2}}}$$

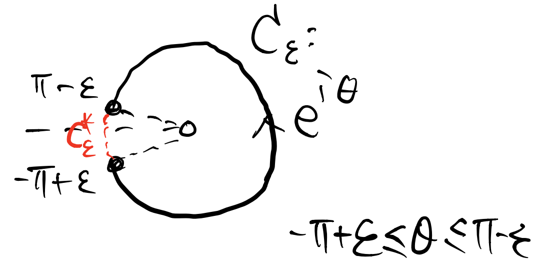
$$= i\frac{\pi}{2} - i\left(-\frac{\pi}{2}\right) = i\pi$$

Therefore $\displaystyle\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = i\pi + i\pi = 2\pi i$

Another way to handle this is by approximation:

Let $C_\varepsilon : z = e^{i\theta}$

$-\pi+\varepsilon \leq \theta \leq \pi - \varepsilon$




Then Principal log can be used and

$$\int_{C_\varepsilon} \frac{dz}{z} = \operatorname{Log} z \Big|_{e^{i(-\pi+\varepsilon)}}^{e^{i(\pi-\varepsilon)}} = i(\pi-\varepsilon) - i(-\pi+\varepsilon)$$

$$= 2\pi i - 2\varepsilon i$$

$$\longrightarrow 2\pi i \text{ as } \varepsilon \to 0.$$

Let $C_\varepsilon^*$ be the gap:

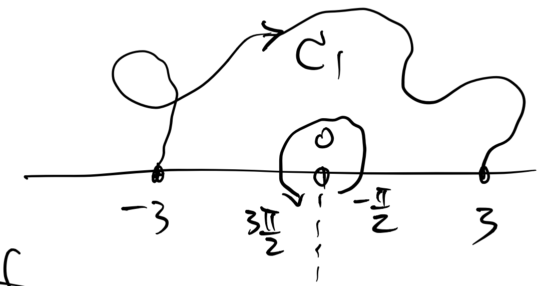
$$\left| \int_{C_\varepsilon} \frac{dz}{z} - \int_C \frac{dz}{z} \right| = \left| \int_{C_\varepsilon^*} \frac{dz}{z} \right| \leq \int_{C_\varepsilon^*} \left|\frac{1}{z}\right| dz$$

$$\leq \text{length } C_\varepsilon^* = 2\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$\therefore \int_C \frac{dz}{z} = \lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{dz}{z} = 2\pi i$$

eg4 : $\int_{C_1} z^{1/2} dz$ where $C_1$ = any contour from $z=-3$ to $z=3$

with $C_1 \setminus \{-3,3\} \subset \{z=x+iy : y>0\}$




Note that the branch of $z^{1/2}$ given by $0<\theta<2\pi$ is not <u>defined</u> at the end point of $C_1$ $(z=3)$.

However, the branch of $z^{1/2}$ given by

$$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

is defined on a domain containing $C_1$.

Then the Thm $\Rightarrow$ (with antiderivative $F(z) = \frac{2}{3} z^{3/2}$, $-\frac{\pi}{2}<\theta<\frac{3\pi}{2}$)

$$\int_{C_1} z^{1/2} dz = \frac{2}{3} z^{3/2} \Big|_{-3=3e^{i\pi}}^{3=3e^{i0}} = \frac{2}{3} \exp\left[\frac{3}{2}\log z\right] \Big|_{3e^{i\pi}}^{3e^{i0}}$$

$$= \frac{2}{3} \exp\left[\frac{3}{2}(\ln|z| + i\theta)\right] \Big|_{3e^{i\pi}}^{3e^{i0}}$$

$$= \frac{2}{3} \exp\left[\frac{3}{2}(\ln 3 + i0)\right] - \frac{2}{3}\exp\left[\frac{3}{2}\ln 3 + i\pi\right]$$

$$= 2\sqrt{3}\,(1 - e^{i\frac{3}{2}\pi}) = 2\sqrt{3}\,(1+i)$$

#

§49 Proof of the Theorem

(a) $\Rightarrow$ (b)

If $C$ is a smooth arc from $z_1$ to $z_2$ & parametrized by $z = z(t)$, $a \le t \le b$.

Then $\frac{d}{dt} F(z(t)) = F'(z(t))\, z'(t) = f(z(t))\, z'(t)$

$$\therefore \int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt$$

$$= F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

If $C$ is piecewise smooth : $C = C_1 + \cdots + C_N$ with $C_i$ smooth arc $\forall i$, joining $z_i$ to $z_{i+1}$.

Then $\int_{C_i} f(z)\,dz = F(z_{i+1}) - F(z_i)$, $\forall i$

$$\Rightarrow \int_C f(z)\,dz = \sum_{i=1}^{N} \int_{C_i} f(z)\,dz = \sum_{i=1}^{N} \left[ F(z_{i+1}) - F(z_i) \right]$$

$$= F(z_{N+1}) - F(z_1)$$

$\therefore \int_C f(z)\,dz$ is independent of path, only depends on end points.

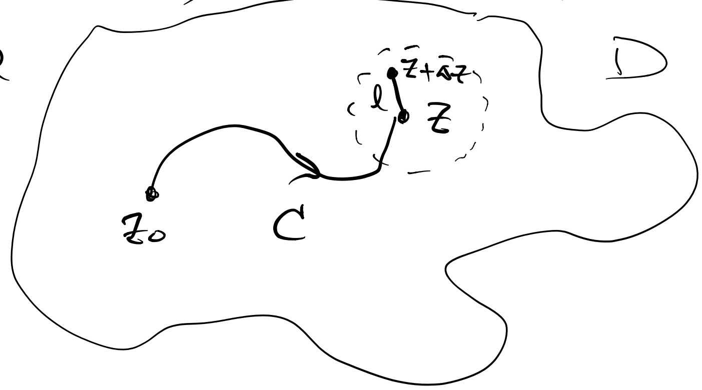
(And the required formula ! )

(b)$\Rightarrow$(a) Fix any $z_0 \in D$.

Then for any $z \in D$, define

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$ which is well-defined because of assumption (b).

For $|\Delta z|$ small, we can choose path as in the figure




to see

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z)\,dz - \int_{z_0}^{z} f(z)\,dz$$

$$= \int_{z}^{z+\Delta z} f(z)\,dz$$

$$\Rightarrow \quad \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s)\,ds - f(z)$$

$$= \int_{z}^{z+\Delta z} \frac{f(s)-f(z)}{\Delta z}\,ds$$

$$\left( \text{Since } \int_{z}^{z+\Delta z} ds = \Delta z \right)$$

Since $f$ is cts, we have

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t.

$$|f(s) - f(z)| < \varepsilon, \quad \forall \ |s - z| < \delta$$

Therefore, for $|\Delta z| < \delta$ & evaluating the integral along the straight line segment $\ell$ between $z$ & $z + \Delta z$, we have

$$\left| \int_z^{z+\Delta z} \frac{f(s) - f(z)}{\Delta z} ds \right| \leq \frac{\varepsilon}{|\Delta z|} \text{ length of } \ell$$

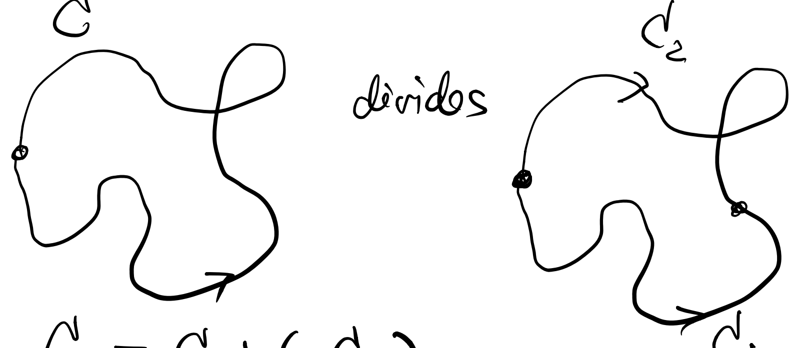
$$= \frac{\varepsilon}{|\Delta z|} |\Delta z| = \varepsilon$$

$\therefore \ \forall \varepsilon > 0, \ \exists \delta > 0$ s.t.

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon, \quad \forall \ |\Delta z| < \delta.$$

$$\therefore \quad F'(z) = f(z), \quad \forall z \in D$$

(b) $\Rightarrow$ (c) (Sketch of proof)

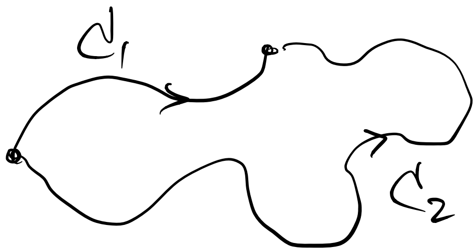



$$C = C_1 + (-C_2)$$

$C_1$ & $C_2$ have same initial & end points

$$\Rightarrow \int_C f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz \underset{\text{by (b)}}{=} 0 .$$

(c) $\Rightarrow$ (b)




Then $C_1 + (-C_2)$ is closed

Assumption (c) $\Rightarrow \int_{C_1+(-C_2)} f\,dz = 0$

$$\Rightarrow \int_{C_1} f\,dz = \int_{C_2} f\,dz \quad \#$$