Def : An arc C given by $z = z(t) = x(t) + iy(t)$, $a \le t \le b$, is called a differentiable arc if $x'(t)$, $y'(t)$ exist and continuous on $[a, b]$.

Note : For differentiable arc $z = z(t) = x(t) + iy(t)$, the function $|z'(t)| = \sqrt{x'^2(t) + y'^2(t)}$ is integrable on $[a, b]$.

Def : The length L of a differentiable arc $z(t) = x(t) + iy(t)$ is defined by

$$L = \int_a^b |z'(t)|\, dt .$$

Prop : The length L of a differentiable arc is independent of the parametrization.

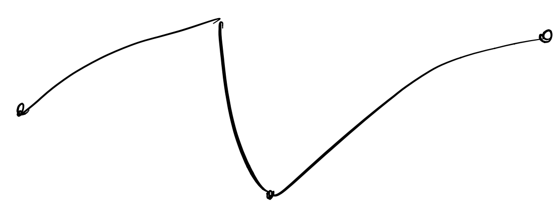
(Pf : in Advanced Calculus)

Note : For differentiable arc $z = z(t)$, if $z'(t) \neq 0$, then $\vec{T} = \dfrac{z'(t)}{|z'(t)|}$ is the unique tangent vector at $z(t)$.

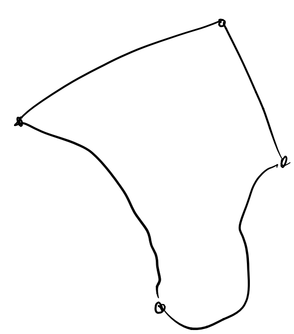
(by identifying cpx numbers as vectors)

Def: (1) An arc is called smooth if $z(t) \in C^1[a,b]$ and $z'(t) \neq 0$, $\forall\, t \in [a,b]$.

(2) A contour, a piecewise smooth arc, is an arc consisting of finite number of smooth arcs joined end to end.



(3) If only the initial & final points are the same, a contour is called a simple closed contour.



Facts: Jordan Curve Theorem

The points on any simply closed contour C are boundary points of 2 distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded.

§44 Contour Integrals

Def: Suppose that a contour C is represented by

$$z = z(t), \quad a \le t \le b$$

with $z_1 = z(a)$ & $z_2 = z(b)$.

(1) A cpx-valued function $f(z)$ is said to be piecewise continuous on C if

$f(z(t))$ is a piecewise cts function on $a \le t \le b$.

(2) The contour integral, a line integral, of f along C in terms of the parameter t is

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

(Think of $dz = z'(t)\,dt$)

Note: The value $\int_C f(z)\,dz$ is indep. of the parameter t.

Pf: Let $t = \phi(\tau)$, $\alpha \le \tau \le \beta$ be a change of parameter.

Then $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

$$= \int_\alpha^\beta f(z(\phi(\tau)))\,z'(\phi(\tau))\,\phi'(\tau)\,d\tau$$

$$= \int_{\alpha}^{\beta} f(Z_1(\tau))\, Z_1'(\tau)\, d\tau$$

where $Z_1(\tau) = z(\phi(\tau))$. #

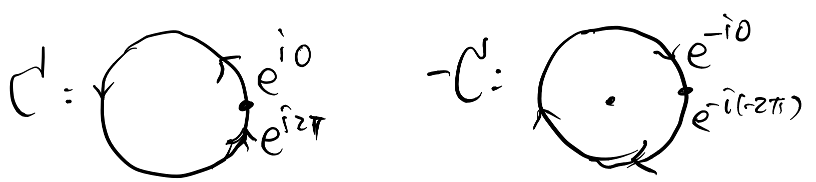
<u>Def</u>: Let C be a contour represented by $z(t)$, $a \le t \le b$.

Then $\underline{\underline{-C}}$ is the contour defined by the reparametrization $z = z(-t)$, $-b \le t \le -a$.

(Same set, but different direction !)

eg: $C: z = e^{it}$, $0 \le t \le 2\pi$

$-C: z = e^{-it}$, $-2\pi \le t \le 0$



<u>Def</u>: (1) If C_1 is a contour from z_1 to z_2, &

C_2 is a contour from z_2 to z_3.

Then <u>sum</u> $C = C_1 + C_2$ is the contour from z_1 to z_3 by first travel from z_1 to z_2 along C_1 and then z_2 to z_3 along C_2

(And we can define $C_1 + C_2 + \cdots + C_N$ similarly)

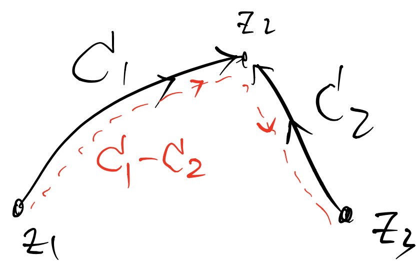
(2) If C_1 is a contour from z_1 to z_2, &

C_2 is a contour from z_3 to z_2.

Then $C_1 + (-C_2)$ is well-defined as in (1) &

is denoted by $C_1 - C_2$.

eg.:



Properties:

(1) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, for constant z_0.

(2) $\int_C [f(z) \pm g(z)]\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$

(3) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(4) $\int_{C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$.

Pf: (1) & (2) are easy.

(3) Let $C: z = z(t)$, $a \le t \le b$

Then $-C: z = z(-t)$, $-b \le t \le -a$.

$$\therefore \int_{-C} f(z)dz = \int_{-b}^{-a} f(z(-t))\left[\frac{d}{dt} z(-t)\right] dt$$

$$= \int_{-b}^{-a} f(z(-t))\left[-z'(-t)\right] dt$$ change of variable $(t \to -t)$

$$= -\int_a^b f(z(t)) z'(t) dt$$

$$= -\int_C f(z)dz \quad \#$$

(4) Let $C_1: z = z_1(t), \ a \le t \le c$ (by choosing suitable parameters)

$C_2: z = z_2(t), \ c \le t \le b$

Then $C_1 + C_2: \ z = \begin{cases} z_1(t), & a \le t \le c \\ z_2(t), & c \le t \le b \end{cases}$

$$\Rightarrow \int_{C_1 + C_2} f(z)dz = \int_a^b f(z(t)) z'(t) dt$$

$$= \int_a^c f(z(t)) z'(t) dt + \int_c^b f(z(t)) z'(t) dt$$

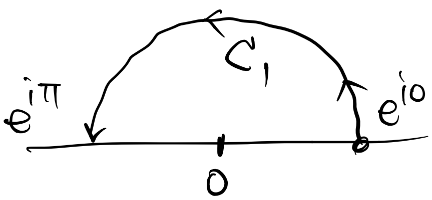
$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

$\#$

§45 Some Examples:

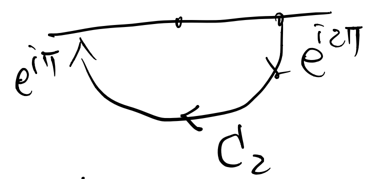
eg 1 (a) Evaluate $\int_{C_1} \frac{dz}{z}$ along $C_1 : z = e^{i\theta}, \ 0 \le \theta \le \pi$

Soln: $\int_{C_1} \frac{dz}{z} = \int_0^{\pi} \frac{d(e^{i\theta})}{e^{i\theta}}$



$$= \int_0^{\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} = i \int_0^{\pi} d\theta = \pi i$$

(b) Evaluate $\int_{-C_2} \frac{dz}{z}$ along C_2:



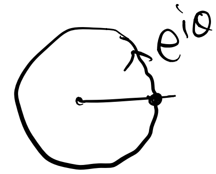
Soln: Note that $-C_2$ can be parametrized by

$$-C_2 : \ z = e^{i\theta}, \quad \pi \le \theta \le 2\pi$$

$$\int_{-C_2} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{d(e^{i\theta})}{e^{i\theta}} = \int_{\pi}^{2\pi} i \, d\theta = i\pi$$

$$\therefore \int_{C_2} \frac{dz}{z} = -\int_{-C_2} \frac{dz}{z} = -i\pi .$$

(c) $\int_C \frac{dz}{z} = ?$ for C:



$0 \le \theta \le 2\pi$

(Ex!)