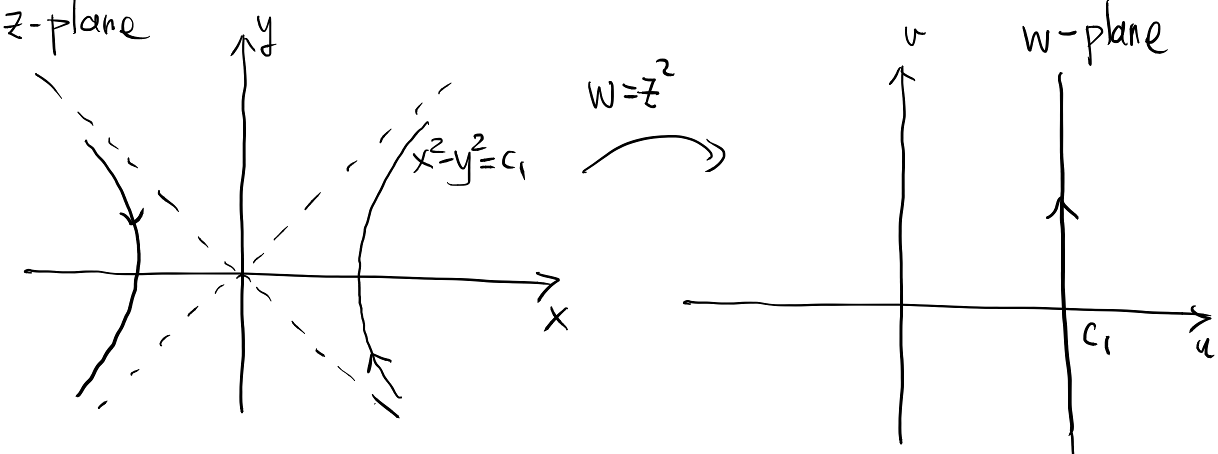
§14 The Mapping $w=z^2$

The $w=z^2$ can be thought of as the transformation

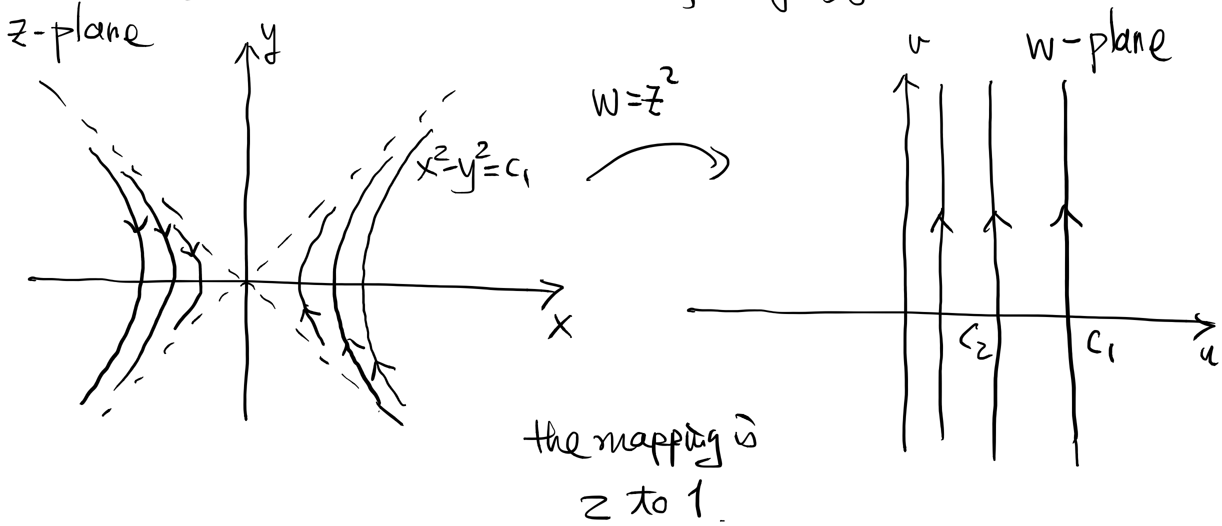
$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$



Consider hyperbola:

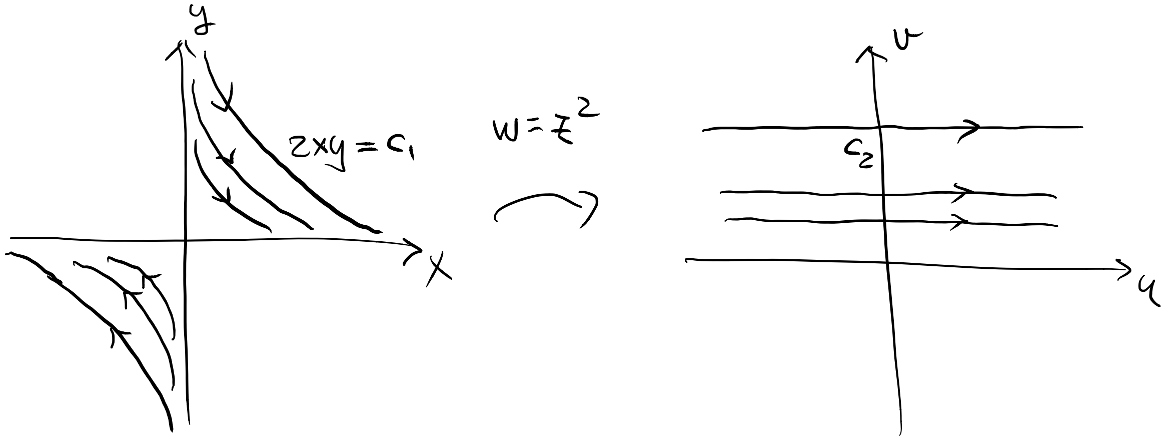
$$x^2 - y^2 = c_1 > 0$$

Allowing c_1 moves, we have the folloing figure:



Ex: What happen for $c_1 < 0$ $(c_1 = 0)$?

Similarly, we can consider $2xy = c_2$ ($c_2 > 0$)



Ex: what happen for $c_2 < 0$ ($c_2 = 0$)?

eg1: The domain $x > 0$, $y > 0$, $xy < 1$

will maps under $w = z^2$ to the horizontal strip $\{0 < v < 2\}$.

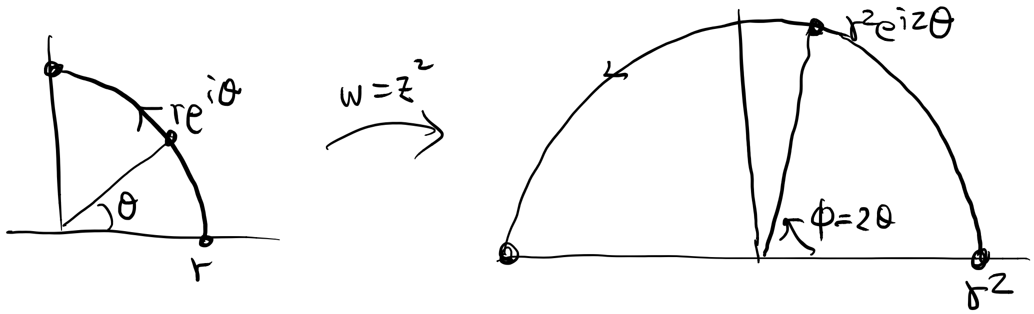
eg2: In exponential form for $w = z^2$:

let $z = re^{i\theta}$, $w = \rho e^{i\phi}$.

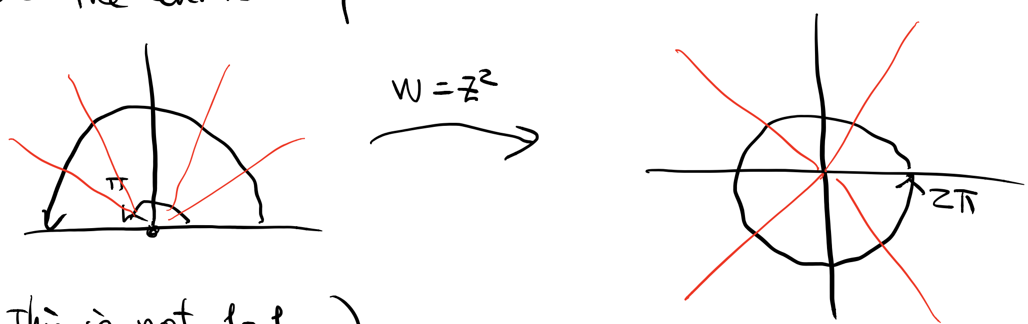
Then $\rho e^{i\phi} = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$\therefore \begin{cases} \rho = r^2 \\ \phi = 2\theta \end{cases}$ this is the polar form of the transformation.

Therefore, one easily sees:



Note $w=z^2$ maps upper half-plane $\{r \geq 0,\ 0 \leq \theta \leq \pi\}$ to the entire w-plane



(This is not 1-1)

§15 Limits

Def: The function $f(z)$ has a limit w_0 as z approaches z_0, denoted by

$$\lim_{z \to z_0} f(z) = w_0,$$

means $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$|f(z) - w_0| < \varepsilon,\ \forall\ 0 < |z - z_0| < \delta.$$

Note: Using mapping representation $(u,v) = f(x,y)$ & note that $|f(z) - w_0| =$ Euclidean distance between the points $f(z)$ & w_0, one sees that the above is equivalent to

$$\lim_{(x,y)\to(x_0,y_0)} \big(u(x,y), v(x,y) \big) = (u_0, v_0)$$

where $w_0 = u_0 + i v_0$, $z_0 = x_0 + i y_0$.

Then we immediately have

Thm If $\lim_{z\to z_0} f(z)$ exists, it is unique.

egs (Ex) (i) $\displaystyle\lim_{z\to 1} \left(i \frac{\bar z}{2} \right) = \frac{i}{2}$

(ii) $\displaystyle\lim_{z\to 0} \frac{z}{\bar z} = \lim_{z\to 0} \frac{re^{i\theta}}{re^{-i\theta}} = \lim_{z\to 0} e^{i2\theta}$

$\displaystyle = \lim_{(x,y)\to 0} (\cos 2\theta, \sin 2\theta)$ doesn't exist.

§16 Theorems on Limits

Thm 1 Suppose that $f(z) = u(x,y) + i v(x,y)$, $z = x+iy$

& $z_0 = x_0 + iy_0$, $w_0 = u_0 + i v_0$

Then
$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \ \& \ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

$\Longleftrightarrow$
$$\lim_{z\to z_0} f(z) = w_0 .$$

Thm 2: Suppose that $\lim_{z\to z_0} f(z) = w_0$, $\lim_{z\to z_0} F(z) = W_0$

Then

(1) $\displaystyle\lim_{z\to z_0} [f(z) \pm F(z)] = w_0 \pm W_0$,

(2) $\lim_{z \to 0} [f(z)F(z)] = w_0 W_0$, &

(3) if $W_0 \neq 0$, $\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$.

§17 Limits involving the point at infinity

Def: The extended complex plane is the union of complex plan $\mathbb{C}$ (= the set of cpx numbers) and the point of infinity $\{\infty\}$.

Notes: (1) We only have one ∞!

(Unlike $\mathbb{R}$ with $\pm\infty$, because we don't have a compatible "inequality" on $\mathbb{C}$.)

(2) The extend cpx plane $\mathbb{C} \cup \{\infty\}$ can be visualized as a sphere via the stereographic projection.