Solution to Homework 6

Sec. 6.1

3. By definition, we have

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Now $f(t) = t$ and $g(t) = e^t$. Then we have the following.

$$\langle f, g \rangle = \int_0^1 te^t dt = \left[te^t\right]_0^1 - \int_0^1 e^t dt = e - (e-1) = 1$$

$$\langle f, f \rangle = \int_0^1 t^2 dt = \left[\frac{1}{3}t^3\right]_0^1 = \frac{1}{3}$$

$$\langle g, g \rangle = \int_0^1 e^{2t} dt = \left[\frac{e^{2t}}{2}\right]_0^1 = \frac{e^2-1}{2}$$

$$\langle f+g, f+g \rangle = \langle f, f \rangle + 2\langle f, g \rangle + \langle g, g \rangle = \frac{1}{3} + 2 + \frac{e^2-1}{2}$$

Hence, we have $\|f\| = \frac{1}{\sqrt{3}}$, $\|g\| = \sqrt{\frac{e^2-1}{2}}$ and $\|f+g\| = \sqrt{\frac{1}{3} + 2 + \frac{e^2-1}{2}}$. To verify the Cauchy-Schwarz inequality, it is easy to check that

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \cdot \langle g, g \rangle$$
$$1 \leq \frac{1}{3} \cdot \frac{e^2-1}{2} \approx 1.0648$$

To verify the triangle inequality, we check the following.

$$\|f+g\| \leq \|f\| + \|g\|$$
$$2.3511 \approx \sqrt{\frac{1}{3} + 2 + \frac{e^2-1}{2}} \leq \frac{1}{\sqrt{3}} + \sqrt{\frac{e^2-1}{2}} \approx 2.3647$$

Hence, both inequalities are satisfied.

8. To check that each of them is not an inner product on the given vector spaces, we need to check that each of them fails some of the conditions to be an inner product.

(a) Note that for $(1,1) \in \mathbb{R}^2$, we have

$$\langle (1,1),(1,1) \rangle = 1 \cdot 1 - 1 \cdot 1 = 0$$

but $(1,1) \neq \mathbf{0}$.

(b) Note that for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, we have

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = \operatorname{tr} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 0$$

but $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \mathbf{0}$.

(c) Note that for $1 \in P(\mathbb{R})$, we have

$$\langle 1, 1 \rangle = \int_0^1 0 \cdot 1 dt = 0$$

but $1 \neq \mathbf{0}$.

9. Let $\beta = \{z_1, z_2, \ldots, z_n\}$ be a basis.

(a) As β is a basis, we can express x as a combination of $\{z_1, z_2, \ldots, z_n\}$.

$$x = a_1 z_1 + a_2 z_2 + \cdots a_n z_n$$

For each $i = 1, 2, \ldots, n$, we have $\langle x, z_i \rangle = 0$, that is, $a_i = 0$. In other words, $x = 0$.

(b) Suppose $\langle x, z \rangle = \langle y, z \rangle$, we have

$$\langle x - y, z \rangle$$

by the linearity of inner product. Then, by (a), we have $x - y = 0$, that is, $x = y$.

10. Suppose x and y are orthogonal vectors in V. That means $\langle x, y \rangle = \langle y, x \rangle = 0$. Then we have the following.

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned}$$

In particular, when $V = \mathbb{R}^2$, let $A, B, C \in \mathbb{R}^2$ be vertices of triangle ΔABC. Taking $x = \overrightarrow{AB}$ and $y = \overrightarrow{BC}$, so $x + y = \overrightarrow{AC}$, by the above, we have

$$\|\overrightarrow{AC}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2.$$

11. The result follows directly from computation.

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= (\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle) \\ &\quad + (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\ &= 2\|x\|^2 + 2\|y\|^2\end{aligned}$$

The equation tells that the sum of the squares of the lengths of the four sides of a parallelogram (right hand side) equals the sum of the squares of the lengths of the two diagonals (left hand side).

12. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V and $a_1, a_2, \ldots, a_k$ be scalars. This means $\langle v_i, v_j\rangle = \langle v_j, v_i\rangle = 0$ for every $i \neq j$. Then we have the following.

$$\begin{aligned}\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 &= \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j\rangle \\ &= \sum_{i=1}^{k} a_i \overline{a_i} \langle v_i, v_i\rangle + \sum_{i\neq j} a_i \overline{a_j} \langle v_i, v_j\rangle \\ &= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2\end{aligned}$$

17. Suppose that $\|T(x)\| = \|x\|$. If $T(x) = T(y)$, then, as T is linear, we have $T(x-y) = 0$. So $\|x-y\| = \|T(x-y)\| = 0$, which means $x = y$. Hence, T is one-to-one.

18. Suppose T is one-to-one, we want to show that $\langle \cdot, \cdot \rangle'$ defines an inner product on V.

    (a) Linearity

$$\begin{aligned}\langle x+z, y\rangle' &= \langle T(x+z), T(y)\rangle \\ &= \langle T(x)+T(z), T(y)\rangle \\ &= \langle T(x), T(y)\rangle + \langle T(z), T(y)\rangle = \langle x, y\rangle' + \langle z, y\rangle'\end{aligned}$$

$$\begin{aligned}\langle cx, y\rangle' &= \langle T(cx), T(y)\rangle \\ &= \langle cT(x), T(y)\rangle \\ &= c\langle T(x), T(y)\rangle = c\langle x, y\rangle'\end{aligned}$$

(b) Conjugate symmetry

$$\begin{aligned}\langle x, y\rangle' &= \langle T(x), T(y)\rangle \\ &= \overline{\langle T(y), T(x)\rangle} = \overline{\langle y, x\rangle'}\end{aligned}$$

(c) Positive-definiteness

$$\langle x, x\rangle' = \langle T(x), T(x)\rangle > 0 \text{ as } x \neq 0 \Rightarrow T(x) \neq 0$$

Conversely, if $\langle\cdot,\cdot\rangle'$ defines an inner product on V. By the positive-definiteness, if $T(x) = 0$, then $\langle x, x\rangle' = \langle T(x), T(x)\rangle = 0$, so $x = 0$. Hence, T is one-to-one.

19. (a) From direct computation, we have

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\ &= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2.\end{aligned}$$

Similarly, we have

$$\|x-y\|^2 = \|x\|^2 - 2\mathrm{Re}\langle x, y\rangle + \|y\|^2.$$

(b) Note that for any x and y, we have the following.

$$\begin{cases}\|x\| \leq \|x-y\| + \|y\| \\ \|y\| \leq \|y-x\| + \|x\|\end{cases}$$

But $\|x-y\| = \|y-x\|$.

$$\begin{cases}\|x\| - \|y\| \leq \|x-y\| \\ \|y\| - \|x\| \leq \|x-y\|\end{cases}$$

Hence, the result follows.

$$\big|\|x\| - \|y\|\big| \leq \|x-y\|$$

21. Note that $A^* = \overline{A^t}$, so $(A^*)^* = A$.

(a) From the above fact, we have the following.

$$A_1^* = \left(\frac{1}{2}(A + A^*)\right)^* = \frac{1}{2}\left(A^* + (A^*)^*\right) = \frac{1}{2}(A^* + A) = A_1$$

$$A_2^* = \left(\frac{1}{2i}(A - A^*)\right)^* = -\frac{1}{2i}\left(A^* - (A^*)^*\right) = -\frac{1}{2i}(A^* - A) = A_2$$

Lastly, we have

$$A_1 + iA_2 = \frac{1}{2}(A + A^*) + i\frac{1}{2i}(A - A^*) = A$$

You can imagine that the *-operation to a matrix is like the complex conjugation to a number. For example, in complex numbers, we have

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}), \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

Similarly, using the adjoint operation, we have

$$A_1 = \frac{1}{2}(A + A^*), \quad A_2 = \frac{1}{2i}(A - A^*)$$

Then it is reasonable to say A_1 is like the "real" part of A and A_2 is like the "imaginary" part of A. Actually, they do share some similar properties.

One easy example is self-adjointness. For complex numbers, if $z = \bar{z}$, we know that z is real. For matrices, if $A = A^*$, A is actually "real" too, but in the sense of *eigenvalues*. In this case, A is called self-adjoint.

(b) Suppose $A = B_1 + iB_2$, $B_1^* = B_1$ and $B_2^* = B_2$. Then we see that

$$A^* = (B_1 + iB_2)^* = B_1^* - iB_2^* = B_1 - iB_2.$$

Solving B_1 and B_2 yields $A_1 = B_1$ and $A_2 = B_2$.

$$B_1 = \frac{1}{2}(A + A^*) = A_1, \quad B_2 = \frac{1}{2i}(A - A^*) = A_2$$