THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT 5000 Analysis I 2015-2016

Suggested Solution to Problem Set 2

1. (a)

$$\begin{aligned} a+b &= 0 \\ (-a)+(a+b) &= (-a)+0 \\ (-a+a)+b &= -a \\ 0+b &= -a \\ b &= -a \end{aligned}$$

(b) Since $(-a)+a=0$, the result in (a) shows that $a=-(-a)$.

(c) Since
$$a+[(-1)(a)] = (1+(-1))a = 0\cdot a = 0,$$
the result in part (a) shows that $(-1)a=-a$.

(d) This part follows directly from part (b) and (c).

(e) From part (c),
$$-(a+b) = (-1)(a+b) = (-1)a+(-1)b = (-a)+(-b).$$

(f)

$$\begin{aligned} (-a)\cdot(-b) &= [(-1)a]\cdot[(-1)b] \\ &= [a(-1)]\cdot[(-1)b] \\ &= a\{(-1)[(-1)b]\} \\ &= a\{[(-1)(-1)]b\} \\ &= a\cdot b. \end{aligned}$$

(g) Note that we need to assume $a\neq 0$. It suffices to show that
$$(-a)\cdot\left(-\frac{1}{a}\right) = 1.$$
From part (c), we have
$$(-a)\cdot\left(-\frac{1}{a}\right) = [(-1)a]\cdot\left[(-1)\left(\frac{1}{a}\right)\right] = a[(-1)(-1)]\left(\frac{1}{a}\right) = a\left(\frac{1}{a}\right) = 1.$$

(h)

$$-\left(\frac{a}{b}\right) = (-1)[a\left(\frac{1}{b}\right)] = [(-1)a]\left(\frac{1}{b}\right) = \frac{(-a)}{b}$$

2.

$$\begin{aligned} a \cdot a &= a \\ a \cdot a + (-a) &= a + (-a) \\ a \cdot a + (-1) \cdot a &= 0 \\ (a + (-1)) \cdot a &= 0. \end{aligned}$$

Thus, $a + (-1) = 0$ or $a = 0$, that is, $a = 1$ or $a = 0$.

3. It suffices to show that

$$\begin{aligned} (ab)\left[\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\right] &= (ab)\left[\left(\frac{1}{b}\right)\left(\frac{1}{a}\right)\right] \\ &= a[b\left(\frac{1}{b}\right)]\left(\frac{1}{a}\right) \\ &= a\left(\frac{1}{a}\right) \\ &= 1. \end{aligned}$$

4. (a) It suffices to show that

$$(b + d) - (a + c) = (b - a) + (d - c) \geq b - a > 0.$$

(b) $0 \leq ac$ is trivial. For the second inequality, we have

$$bd - bc = b(d - c) \in \mathbb{P} \cup \{0\}$$

and

$$bc - ac = (b - a)c \in \mathbb{P} \cup \{0\}.$$

Thus, $0 \leq ac \leq bd$.

5. (a) The equality follows from the definition of the inverses. We need show that $\frac{1}{a} \in \mathbb{P}$. By means of a contradiction, we suppose $\frac{1}{a} \notin \mathbb{P}$. Since $\frac{1}{a} \neq 0$ (otherwise $a \cdot \frac{1}{a} = 0 \neq 1$), we must have $-\frac{1}{a} \in \mathbb{P}$ by the Trichotomy property. Since $\mathbb{P}$ is closed under multiplications, we have

$$-1 = a \cdot \left(-\frac{1}{a}\right) \in \mathbb{P}.$$

It is a contradiction and hence $\frac{1}{a} \in \mathbb{P}$.

(b) It suffices to show that

$$b - \frac{1}{2}(a+b) = \frac{1}{2}(a+b) - a = \frac{1}{2}(b-a) \in \mathbb{P}$$

since $\frac{1}{2}, b-a \in \mathbb{P}$.

6. (a) We use Mathematical Induction. Both the first step and the induction step are trivial.

   (b) Suppose there exists $n, m \in \mathbb{N}$ such that $2n = 2m+1$. Clearly $n > m$ and so $n - m \in \mathbb{N}$. However, $n - m = \frac{1}{2} \notin \mathbb{N}$ and it is a contradiction.

7. Clearly S_1 is bounded below by 0. Moreover, for each $\varepsilon > 0$,

$$0 + \varepsilon < \frac{\varepsilon}{2} \in S_1,$$

so 0 is the greatest lower bound and $\inf S_1 = 0$.

As for an upper bound of S_1, it does not exist and it follows from the Archimedian Property.

8. $\inf S_2 = 0$ and S_2 is not bounded above. The same arguments as in the last problem continues to hold.

9. Clearly, S_3 is bounded above by 1 and $1 \in S_3$. So $\sup S_3 = 1$. For the lower boundedness, clearly, S_3 is bounded below by 0 and given $\varepsilon > 0$, the Archimedian Property shows the existence of $n \in \mathbb{N}$ such that

$$0 + \varepsilon < \frac{1}{n}.$$

Hence, $\inf S_3 = 0$.

10. Clearly, S_4 is bounded below by 0 and above by 2. Since $0, 2 \in S_4 = \{0, 2\}$, $\inf S_4 = 0$ and $\sup S_4 = 2$.

11. Since S_5 is bounded below by 0, we must have $\inf S_5 \geq 0$. Also, $S_3 \subset S_5$ shows that $\inf S_5 \leq \inf S_3 = 0$. Hence, $\inf S_5 = 0$.

12. Denote $t = \sup\{-s : s \in S\}$. We need to show that $-t$ is a lower bound of S and t is the maximal among the lower bounds. Now, by the definition of an upper bound

$$-s \leq t \quad \forall s \in S,$$

that is,

$$s \geq -t \quad s \in S$$

and so $-t$ is a lower bound. Given $\varepsilon > 0$, since t is the least upper bound, there exists $s \in S$ such that

$$t - \varepsilon < -s,$$

or,

$$s < -t + \varepsilon.$$

This shows that $-t$ is maximal among the lower bounds and the proof is completed.

13. We will prove the statement:

Given a uniformly bounded family of non-empty sets $\{A_\alpha\}_{\alpha \in I}$, where I is an index set,

$$\sup\left(\bigcup_{\alpha \in I} A_\alpha\right) = \sup\{\sup A_\alpha : \alpha \in I\}.$$

Here, the family of sets is uniformly bounded if there exists $M_1, M_2 \in \mathbb{R}$ such that $M_1 \le a \le M_2$ for all $a \in \cup_{\alpha \in I} A_\alpha$. Denote $\beta_\alpha = \sup A_\alpha$ and $\beta = \sup \cup_{\alpha \in I} A_\alpha$. First, since $A_\alpha \subset \cup_{\alpha \in I} A_\alpha$, $\beta_\alpha \le \beta$ for all $\alpha \in I$. In particular, $\sup_{\alpha \in I} \beta_\alpha \le \beta$. Now, we show that β is the minimal among the upper bounds. By the definition of β, given $\varepsilon > 0$, there exists $a \in \cup_{\alpha \in I} A_\alpha$ such that $\beta - \varepsilon < a$. Without loss of generality, we may assume $a \in A_{\alpha_0}$ for some $\alpha_0 \in I$ and so

$$\beta - \varepsilon < a \le \beta_{\alpha_0}.$$

This shows that β is the minimal among the upper bounds and the proof is completed.

Note that the conclusion is false if we do not add the uniformly bounded condition in the assumption for which the existence of β is not a must. Consider the family of sets $\{\{x\}\}_{x \in \mathbb{R}}$, then each of the set $\{x\}$ is bounded whereas the family of sets is not bounded as a whole.

14. Set $A = S$ and $B = \{u\}$ in the previous problem and the result follows.

15. We use Mathematical Induction. The first step is trivial and the induction step follows from the last problem.

16. Clearly, $-1, 1 \in S$ and S is bounded above and below by 1 and -1 respectively and hence $\inf S = -1$ and $\sup S = 1$.

17. We only prove the first equality. From the definitions, $a \le \sup A$ and $b \le \sup B$ for all $a \in A, b \in B$. Hence, $a + b \le \sup A + \sup B$ for all $a \in A, b \in B$. This shows that $\sup(A + B) \le \sup A + \sup B$. To show that

it is minimal among the upper bounds, given $\varepsilon > 0$ there exists $a_\varepsilon \in A$ and $b_\varepsilon \in B$ such that

$$\sup A - \frac{\varepsilon}{2} < a_\varepsilon$$

and

$$\sup B - \frac{\varepsilon}{2} < b_\varepsilon.$$

Summing them together,

$$(\sup A + \sup B) - \varepsilon < a_\varepsilon + b_\varepsilon \in A + B$$

and the result follows.

18. Using the Mathematical Induction, we can easily show that $2^n > n$ for any $n \in \mathbb{N}$. The result then follows from an application of the Archimedian Property.

19. Note that $x < ru < y$ is equivalent to $\frac{x}{u} < r < \frac{y}{u}$ and so the statement is equivalent to the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ (please refer to Chapter 2 for the proof).

20. Note that $\frac{a}{n} < b$ is equivalent to $\frac{a}{b} < n$ and so that it follows from the Archimedian Property.

21. Clearly, $0 \in I_n$ for any $n \in \mathbb{N}$ and so $0 \in \cap_{n=1}^{\infty}$. Now, given any $0 < \varepsilon < 1$, the Archimedian Property shows that there exists $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \varepsilon$, meaning $\varepsilon \notin I_{n_\varepsilon}$ and so $\varepsilon \notin \cap_{n=1}^{\infty} I_n$.

22. The same argument holds except $0 \notin \cap_{n=1}^{\infty} J_n$.

23. The correct statement is $\cap_{n=1}^{\infty} K_n = \emptyset$. Now, given any $x \in \mathbb{R}$, the Archimedian Property shows the existence of $n_x \in \mathbb{N}$ such that $x < n_x$. This shows that $x \notin K_n$ and hence $x \notin \cap_{n=1}^{\infty} K_n$. Since $x \in \mathbb{R}$ is arbitrary, we conclude that $\cap_{n=1}^{\infty} K_n = \emptyset$.

24. The first statement is trivial and the second statement follows from the nested property that $I_{2n} \subseteq I_{2n-1}$ for $n \in \mathbb{N}$.