1 (i) $\phi, X \in \mathcal{T}$

(ii) There are 5 elements in $\mathcal{T}$.

So number of possible unions $= C^5_1 + C^5_2 + \cdots + C^5_5$

$= 31$

ϕ, $\{b\}$, $\{a,b\}$, $\{b,c\}$, X, $\phi \cup \{b\}$, $\phi \cup \{a,b\}$, $\phi \cup \{b,c\}$, $\phi \cup X$

$\{b\} \cup \{a,b\}$, $\{b\} \cup \{b,c\}$, $\{b\} \cup X$, $\{a,b\} \cup \{b,c\}$, $\{a,b\} \cup X$,

$\{b,c\} \cup X$, $\phi \cup \{b\} \cup \{a,b\}$, $\phi \cup \{b\} \cup \{b,c\}$, $\phi \cup \{b\} \cup X$,

$\phi \cup \{a,b\} \cup \{b,c\}$, $\phi \cup \{a,b\} \cup X$, $\phi \cup \{b,c\} \cup X$,

$\{b\} \cup \{a,b\} \cup \{b,c\}$, $\{b\} \cup \{a,b\} \cup X$, $\{a,b\} \cup \{b,c\} \cup X$,

$\phi \cup \{b\} \cup \{a,b\} \cup \{b,c\}$, $\phi \cup \{b\} \cup \{a,b\} \cup X$,

$\phi \cup \{b\} \cup \{b,c\} \cup X$, $\phi \cup \{a,b\} \cup \{b,c\} \cup X$,

$\{b\} \cup \{a,b\} \cup \{b,c\} \cup X$, $\phi \cup \{b\} \cup \{a,b\} \cup \{b,c\} \cup X$

They are all in $\mathcal{T}$.

(Remark: Since X is a finite set, any union is a finite union.)

(iii) Repeat the above list but changing $\cup$ to be $\cap$ and check everyone is in $\mathcal{T}$.

2 (i) $X \setminus \phi = X \Rightarrow \phi \in \mathcal{T}$

$X \setminus X = \phi$ which is a finite set $\Rightarrow X \in \mathcal{T}$

(ii) Let $U_\alpha \in \mathcal{T}$ where $\alpha \in I$, I is an index set

If all U_α are empty, then $\bigcup_{\alpha \in I} U_\alpha = \phi$ and

$X \setminus \bigcup_{\alpha \in I} U_\alpha = X \setminus \phi = X \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

If there exists $U_{\alpha_0} \neq \phi$, then

$\bigcup_{\alpha \in I} U_\alpha \supseteq U_{\alpha_0}$

$X \setminus \bigcup_{\alpha \in I} U_\alpha \subseteq X \setminus U_{\alpha_0}$ and $X \setminus U_{\alpha_0}$ is a finite set

$\therefore X \setminus \bigcup_{\alpha \in I} U_\alpha$ is a finite set $\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

(iii) Let $U_1, U_2, \cdots, U_n \in \mathcal{J}$

If there exists U_i which is ϕ, then $\bigcap_{i=1}^{n} U_i = \phi$

and $X \setminus \bigcap_{i=1}^{n} U_i = X \setminus \phi = X$

$\therefore \bigcap_{i=1}^{n} U_i \in \mathcal{J}$

If all $U_i \neq \phi$, then $X \setminus U_i$ is a finite set.

and

$$X \setminus \bigcap_{i=1}^{n} U_i = (X \setminus U_1) \cup (X \setminus U_2) \cup \cdots \cup (X \setminus U_n)$$

which is a finite set.

$\therefore \bigcap_{i=1}^{n} U_i \in \mathcal{J}$

3 (i) $\phi \in \mathcal{J}$ which is trivial as there is no element in ϕ.

Let $x \in X$, by definition of $\mathcal{B}$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$.

$\therefore X \in \mathcal{J}$.

(ii) Let $U_\alpha \in \mathcal{J}$, $\alpha \in I$ and I is an index set.

Let $x \in$, then $x \in U_{\alpha_0}$ for some fixed $\alpha_0 \in I$

Since $U_{\alpha_0} \in \mathcal{J}$, there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$$

$\therefore \bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$.

(iii) Let U_1, U_2 and let $x \in U_1 \cap U_2$

By definition of $\mathcal{B}$, there exists B_i

$x \in B_i \subseteq U_i$ for all $i = 1, 2$

then there exists $B' \in \mathcal{B}$ such that

$x \in B' \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$

$\therefore \quad U_1 \cap U_2 \in \mathcal{J}$

In general, by Mathematical Induction, it can be proved that $U_1, \ldots, U_n \in \mathcal{J} \Rightarrow U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{J}$

4 (i) Let $x \in \mathbb{R}$,

we have $x \in (x-1, x+1) \in \mathcal{B}$

(ii) If $x \in (a,b) \cap (c,d)$, take $\delta = \frac{1}{2}\min\{|x-a|, |x-b|, |x-c|, |x-d|\}$ and consider $(x-\delta, x+\delta)$, then $x \in (x-\delta, x+\delta) \subseteq (x-1, x+1)$.

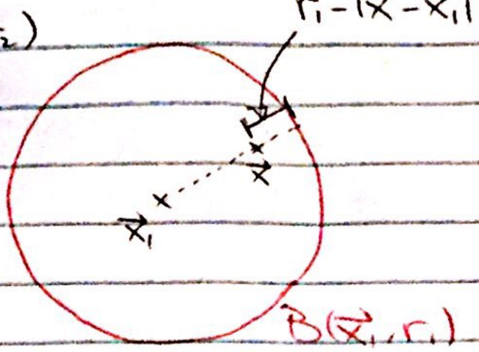
5 (i) Let $\vec{x} \in \mathbb{R}^2$,

we have $\vec{x} \in B(\vec{x}, 1) \in \mathcal{B}$

(ii) If $\vec{x} \in B(\vec{x}_1, r_1) \cap B(\vec{x}_2, r_2)$,

take $\delta = \frac{1}{2}\min\{ r_1 - |\vec{x} - \vec{x}_1|, r_2 - |\vec{x} - \vec{x}_2| \}$

then $\vec{x} \in B(\vec{x}, \delta) \subseteq B(\vec{x}_1, r_1) \cap B(\vec{x}_2, r_2)$



6 Let $\{x_n\} \subseteq X$ be a convergent sequence and $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} x_n = v$, where

Since X is Hausdorff, there exist open subsets U and V such that $u \in U$, $v \in V$ and $U \cap V = \phi$.

$\lim_{n\to\infty} x_n = u \Rightarrow \exists N_1 \in \mathbb{N}$ st. $\forall n \geq N_1$, $x_n \in U$

$\lim_{n\to\infty} x_n = v \Rightarrow \exists N_2 \in \mathbb{N}$ st. $\forall n \geq N_2$, $x_n \in V$

but consider $N = \max\{N_1, N_2\}$, then $x_N \in U \cap V$ which contradicts to $U \cap V = \phi$.

$\therefore u = v$.

7 (i) By definition, $d((a_1,a_2,a_3),(b_1,b_2,b_3))$
$= \text{number of distinct components}$
≥ 0

If $d((a_1,a_2,a_3),(b_1,b_2,b_3)) = 0$
then every component is the same.
$\therefore (a_1,a_2,a_3) = (b_1,b_2,b_3)$

(ii) $d((a_1,a_2,a_3),(b_1,b_2,b_3)) = d((a_1,a_2,a_3),(b_1,b_2,b_3))$

(iii) Let $(a_1,a_2,a_3), (b_1,b_2,b_3), (c_1,c_2,c_3) \in X$.
Let $\alpha \subseteq \{1,2,3\}$ where α is the collection of distinct components of (a_1,a_2,a_3) and (b_1,b_2,b_3)
Similarly, let $\beta \subseteq \{1,2,3\}$ where β is the collection of distinct components of (b_1,b_2,b_3) and (c_1,c_2,c_3).
Then the collection of distinct components of (a_1,a_2,a_3) and (c_1,c_2,c_3) must be a subset of $\alpha \cup \beta$

$$\begin{aligned}
\therefore d((a_1,a_2,a_3),(c_1,c_2,c_3)) &\leq |\alpha \cup \beta| \\
&\leq |\alpha| + |\beta| \\
&\leq d((a_1,a_2,a_3),(b_1,b_2,b_3)) \\
&\quad + d((b_1,b_2,b_3),(c_1,c_2,c_3))
\end{aligned}$$

8 (a) (i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$.
(ii) $d(x,y) = d(y,x)$
(iii) Let $x,y,z \in X$

If $x = z$, then $d(x,z) = 0 \leq d(x,y) + d(y,z)$
If $x \neq z$, then at least we have $y \neq x$ or $y \neq z$.
$\therefore d(x,z) = 1 \leq d(x,y) + d(y,z)$

(b) Let $x \in X$, $B(x,\frac{1}{2}) \in \mathcal{B}$.
But $B(x,\frac{1}{2}) = \{x\}$.
$\therefore \{a\}, \{b\}, \{c\} \subseteq \mathcal{T}$
By definition of topology, any unions of elements in $\mathcal{T}$ again in $\mathcal{T}$, so $\mathcal{T}$ = all subsets of X.

9 (i) $\|(x_1, x_2, \cdots, x_n)\| \geq 0$

and if $\|(x_1, x_2, \cdots, x_n)\| = 0$, then $|x_1| = |x_2| = \cdots = |x_n| = 0$

i.e. $x_1 = x_2 = \cdots = x_n = 0$

(ii) If $c \in \mathbb{R}$

then $\|c(x_1, \cdots, x_n)\| = \max\{|cx_1|, |cx_2|, \cdots, |cx_n|\}$

$= |c| \max\{|x_1|, |x_2|, \cdots, |x_n|\}$

$= |c| \, \|(x_1, \cdots, x_n)\|$.

(iii) Let $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$

$\|(x_1, x_2, \cdots, x_n\| + \|(y_1, \cdots, \cdots, y_n\|$

$= \max\{|x_1|, \cdots, |x_n|\} + \max\{|y_1|, \cdots, |y_n|\}$

$\geq |x_i| + |y_i|$ for all $i = 1, 2, \cdots, n$

$\geq |x_i + y_i|$ ($\triangle$ ineq.)

$\therefore \|(x_1, \cdots, x_n)\| + \|(y_1, \cdots, y_n)\| \geq \|(x_1 + y_1, \cdots, x_n + y_n)\|$.

10 No, consider $f: [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } a < x \leq b \end{cases}$$

Then $\int_a^b f = 0$ but f is NOT the zero function.

i.e. $\exists f \neq 0$ but $\|f\| = 0$

11 (i) $\|f\| = \sqrt{\int_a^b |f|^2}$

≥ 0 $\left(|f|^2 \geq 0 \Rightarrow \int_a^b |f|^2 \geq 0\right)$

Since f is continuous, $|f|^2$ is continuous.

If $\|f\| = 0$, then $\int_a^b |f|^2 = 0$

$\therefore |f|^2 = 0 \Rightarrow f = 0$ (why?)

(ii) $\|cf\| = \sqrt{\int_a^b |cf|^2}$

$= \sqrt{|c|^2 \int_a^b |f|^2}$

$= |c| \sqrt{\int_a^b |f|^2}$

$= |c| \|f\|$.

(iii) Check : Minkowski Inequality.