THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT 5000 Analysis I 2015-2016

Suggested Solution to Problem Set 1

1. If E or F is empty, then the statement is trivial. Therefore, we assume both E and F are non-empty. Let $y \in f(E \cup F)$, then

$$\begin{aligned} y \in f(E \cup F) &\iff \exists x \in E \cup F \text{ such that } y = f(x) \\ &\iff y = f(x) \text{ for some } (x \in E \text{ or } x \in F) \\ &\iff y \in f(E) \text{ or } y \in f(F) \\ &\iff y \in f(E) \cup f(F). \end{aligned}$$

Let $v \in f(E \cap F)$, then

$$\begin{aligned} v \in f(E \cap F) &\implies \exists u \in E \cap F \text{ such that } v = f(u) \\ &\implies v = f(u) \text{ for some } (u \in E \text{ and } u \in F) \\ &\implies v \in f(E) \text{ and } v \in f(F) \\ &\implies v \in f(E) \cap f(F). \end{aligned}$$

Note that the reverse side from the third to second line is false in general.

2. First we assume $f^{-1}(G \cup H)$ is non-empty. Let $x \in f^{-1}(G \cup H)$, then

$$\begin{aligned} x \in f^{-1}(G \cup H) &\iff \exists y \in G \cup H \text{ such that } y = f(x) \\ &\iff \exists (y \in G \text{ or } y \in H) \text{ such that } y = f(x) \\ &\iff x \in f^{-1}(G) \text{ or } x \in f^{-1}(H) \\ &\iff x \in f^{-1}(G) \cup f^{-1}(H). \end{aligned}$$

If $f^{-1}(G \cup H)$ is empty, then since clearly $f^{-1}(G), f^{-1}(H) \subseteq f^{-1}(G \cup H)$, $f^{-1}(G)$ and $f^{-1}(H)$ are both empty and the hence the equality holds. Next, again, we assume $f^{-1}(G) \cap f^{-1}(H)$ is non-empty. Let $u \in f^{-1}(G) \cap f^{-1}(H)$, then

$$\begin{aligned} &u \in f^{-1}(G) \cap f^{-1}(H) \\ &\implies u \in f^{-1}(G) \text{ and } u \in f^{-1}(H) \\ &\implies \exists v_1 \in G \text{ and } v_2 \in H \text{ such that } v_1 = f(u) \text{ and } v_2 = f(u). \end{aligned}$$

By the definition of a function, we must have that $f(u)$ is a single element and hence $v_1 = v_2$. Therefore, $v_1 = v_2 \in G \cap H$ and $u \in f^{-1}(G \cap H)$. This shows that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$. Conversely, since we clearly have

$$f^{-1}(G \cap H) \subseteq f^{-1}(G), f^{-1}(H),$$

we get

$$f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H).$$

Altogether, $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$. Finally, when $f^{-1}(G) \cap f^{-1}(H)$ is empty, since $f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H)$, $f^{-1}(G \cap H)$ must also be empty and the equality holds.

3. (a) We assume E is non-empty; otherwise, the statement is trivial. Let $u \in E$, then $f(u) \in f(E)$ and so $u \in f^{-1}(f(E))$ by definition. This shows that $E \subseteq f^{-1}(f(E))$. Next, we suppose $f^{-1}(f(E))$ is non-empty and let $x \in f^{-1}(f(E))$; otherwise, the first part shows that E is also empty and the result follows. Now,

$$x \in f^{-1}(f(E)) \Longrightarrow \exists y \in f(E) \text{ such that } y = f(x).$$

Since $y \in f(E)$, there exists $x' \in E$ such that $f(x) = y = f(x')$. The injectivity of f shows that $x = x' \in E$. Therefore, if f is injective, then $f^{-1}(f(E)) = E$.

To show that the injectivity of f is essential, consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Let $E = \{x \in \mathbb{R} : x \geq 0\}$, then $E \neq \mathbb{R} = f^{-1}(f(E))$.

(b) We assume H is non-empty; otherwise, the statement is trivial. We first assume $f(f^{-1}(H))$ is non-empty (which is indeed the case if we assume f is surjective and H is non-empty). Let $v \in f(f^{-1}(H))$, then $v = f(u)$ for some $u \in f^{-1}(H)$, or $v = f(u) \in H$. Thus, $f(f^{-1}(H)) \subseteq H$ in general. Now, let $y \in H$. The surjectivity of f shows that there exists $x \in A$ such that $y = f(x)$, or $x \in f^{-1}(H)$. Therefore, $y = f(x) \in f(f^{-1}(H))$ and we have shown that $H \subseteq f(f^{-1}(H))$.

To show that the surjectivity is essential, we consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Let $H = \{x \in \mathbb{R} : x < 0\}$, then $H \neq \emptyset = f(f^{-1}(H))$.

4. Note that because f is an injection, the sightly modified function $g : D(f) \to f(A)$ with $g(x) = f(x)$ for $x \in D(f)$ is a bijection and g^{-1} is well-defined. Therefore, in the first part of the question, f^{-1} is actually the g we have just defined.

(a) It follows from the definition of the inverse "f^{-1}" as explained at the beginning.

(b) We need to show that f^{-1} is both injective and surjective.

To see that f^{-1} is injective, let $y_1, y_2 \in B$ and $f^{-1}(y_1) = f^{-1}(y_2)$. Applying the function f on both sides and recalling that $f \circ f^{-1}$ is the identity map, we get $y_1 = y_2$. This shows that f^{-1} is injective. For surjectivity, let $x \in A$. Considering the identity map $f^{-1} \circ f$, we get

$$f^{-1}(f(x)) = x.$$

Clearly, $f(x) \in B$ and hence f^{-1} is surjective.

5. We first show that $g \circ f$ is injective. Let $x_1, x_2 \in A$ and $g \circ f(x_1) = g \circ f(x_2)$. The injectivity of g shows that $f(x_1) = f(x_2)$ and then the injectivity of f shows that $x_1 = x_2$. This shows that $g \circ f$ is injective.

   To see that $g \circ f$ is surjective, let $w \in C$. The surjectivity of g shows that there exists $y \in B$ such that $g(y) = w$ and then the surjectivity of f shows that there exists $x \in A$ such that $f(x) = y$, or $g \circ f(x) = w$. This shows that $g \circ f$ is surjective.

6. (a) Let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. Applying the function g on both sides, we get $g \circ f(x_1) = g \circ f(x_2)$ and the injectivity of $g \circ f$ shows that $x_1 = x_2$.

   (b) Let $w \in C$. The surjectivity of $g \circ f$ shows that existence of $x \in A$ such that $g \circ f(x) = w$, or $g(f(x)) = w$. Since $f(x) \in B = D(g)$, the result follows.

7. Note that the fact that f is a bijection follows from the last question. We first claim that $D(f) = R(g)$ and $D(g) = R(f)$. Now, from the equations, we see that they are possible only if $R(f) \subseteq D(g)$ and $R(g) \subseteq D(f)$. Also, from the equations, for $x \in D(f)$ and $y \in D(g)$, we have

$$x = g \circ f(x) = g(f(x)) \in R(g) \qquad y = f \circ g(y) = f(g(y)) \in R(f).$$

   Hence, our claim holds. Finally, from the commutativity of functions and the identities $f^{-1} \circ f(x) = x$ for $x \in D(f)$ and $f \circ g(y) = y$ for $y \in D(g) = R(f)$, we get

$$\begin{aligned} f \circ g(y) &= y \\ f^{-1}(f \circ g(y)) &= f^{-1}(y) \\ g(y) &= f^{-1}(y). \end{aligned}$$

8. Let $P(n)$ be the statement

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}.$$

We will prove the statement by the Mathematical Induction.

For $n = 1$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}.$$

Hence, the statement is true for $n = 1$.

Suppose $P(k)$ is true; we need to show that $P(k+1)$ is true.

For $n = k+1$,

$$\begin{aligned}
&\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}} + \frac{1}{\sqrt{k+2}} \\
&> \sqrt{k+1} + \frac{1}{\sqrt{k+2}} \\
&= \sqrt{k+2} - \frac{1}{\sqrt{k+1}+\sqrt{k+2}} + \frac{1}{\sqrt{k+2}} \\
&> \sqrt{k+2}.
\end{aligned}$$

Hence, $P(k+1)$ is true and the statement is true for all $n \in \mathbb{N}$ by the principle of the Mathematical Induction.

9. Let $S(n) = P(n + n_0 - 1)$, then the assumptions give

$$\text{The statement } S(1) \text{ is true;}$$

$$\text{For all } k \geq 1, \text{ the truth of } S(k) \text{ implies the truth of } S(k+1).$$

An application of the usual Mathematical Induction on $S(n), n \geq 1$ gives the desired conclusion.

10. We modify the proof for the usual Mathematical Induction.

Suppose $S \neq \mathbb{N}$, then $\mathbb{N} \setminus S$ is non-empty. By the Well-ordering principle, there exists $m \in \mathbb{N} \setminus S$ such that m is the least element in $\mathbb{N} \setminus S$. By the first assumption, $m \neq 1$. Since m is the minimum of $\mathbb{N} \setminus S$, $\{k \in \mathbb{N} : k < m\} \subseteq S$. The second assumption gives $m \in S$ which is a contradiction. Therefore, we must have $S = \mathbb{N}$.

11. We modify the proof for the usual Mathematical Induction.

Suppose $S \neq \mathbb{N}$, then $\mathbb{N} \setminus S$ is non-empty. By the Well-ordering principle, there exists $m \in \mathbb{N} \setminus S$ such that m is the least element in $\mathbb{N} \setminus S$. By the first assumption, $m \neq 2^k$ for any $k \in \mathbb{N}$, say, $m < 2^{k'}$ for some $k' \in \mathbb{N}$. By the second assumption, we must have $m \in S$ which is a contradiction. Therefore, we must have $S = N$.

12. Define the map $f : S \to \mathbb{N}$ by $f(n) = n - 2015$. It is clear that f is bijection and the result follows.

13. We first prove the statement under the addition assumption that $S \cap T$ is empty. By the definition of countably infinity, there exists bijections $f_S : S \to \mathbb{N}$ and $f_T : T \to \mathbb{N}$. Define the function $f : S \cup T \to \mathbb{N}$ such that

$$f(x) = \begin{cases} 2f_S(x) & \text{if } x \in S, \\ 2f_T(x) + 1 & \text{if } x \in T. \end{cases}$$

It is clear that f is a bijection and hence $S \cup T$ is countably infinite.

In general, consider $S \cup T = (S \setminus T) \cup (S \cap T) \cup (T \setminus T)$. The decomposed sets are mutually distinct. Since it is also clearly that the union of two finite sets is finite and the union of a finite set and a countably infinite set is countably infinite, the result follows.

14. Let $f : S \to \mathbb{N}$ be a bijection. Since $S \cap T \subseteq T$, $S \cap T$ is finite, say, $0 \neq |S \cap T| = N \in \mathbb{N}$ (the case $|S \cap T| = 0$ is trivial). Write $S \cap T = \{s_1, s_2, \cdots, s_N\}$. Let $M = \max f(S \cap T)$. We define the function $g : S \setminus T \to \mathbb{N}$ by

$$g(s) = \begin{cases} f(s) & \text{if } s \in f^{-1}(\{n \in \mathbb{N} : n \leq M\}), \\ s_k & \text{if } f^{-1}(s) = M + k, 1 \leq k \leq N, \\ f(s) - N & \text{otherwise.} \end{cases}$$

It is easy to see that g is a bijection and hence $S \setminus T$ is countably infinite.

15. Consider the function $g : S \to R(f)$ where $g(s) = f(s)$ for $s \in S$. Then g is a bijection and hence $R(f)$ is an infinite set. Since $R(f) \subseteq T$, T is also an infinite set.

16. Yes, S is an infinite set. Since f is a surjection, for each $t \in T$, $f^{-1}(\{t\})$ is non-empty. By the axiom of choice, for each $t \in T$, we can pick an $s_t \in f^{-1}(\{t\})$. Define $S_T = \{s_t\}_{t \in T}$, then $S_T \subseteq S$ and the function $f : S_T \to T$ defined by $f(s_t) = t$ is a bijection. Hence, S_T is an infinite set and so S is also an infinite set.

17. (a) $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, S\}$ and hence $|\mathcal{P}(S)| = 2^3 = 8$.

    (b) For $n = 1$, the statement is trivially true.

    Suppose the statement is true for $n = k$, we need to show that the statement holds for $n = k + 1$.

For $n = k + 1$, pick an element $s \in S$, then $|S \setminus \{s\}| = k$. The assumption shows that $|\mathcal{P}(S \setminus \{s\})| = 2^k$. Now,

$$\begin{aligned}
\mathcal{P}(S) &= \{A \in \mathcal{P}(S) : s \in A\} \cup \{A \in \mathcal{P}(S) : s \notin A\} \\
&= \{A \in \mathcal{P}(S) : s \in A\} \cup \mathcal{P}(S \setminus \{s\}) \\
&= \{A \cup \{s\} : A \in \mathcal{P}(S \setminus \{s\})\} \cup \mathcal{P}(S \setminus \{s\}).
\end{aligned}$$

Note that union is disjoint and hence

$$|\mathcal{P}(S)| = 2|\mathcal{P}(S \setminus \{s\})| = 2^{k+1}.$$

The statement holds for $n = k + 1$ and the desired conclusion follows from the Mathematical Induction.