Lecture 5

3.4 Ergodicity

Definition 3.2 (Ergodicity)**.** *A MPS $(X, \mathscr{B}, m, T)$ is said to be ergodic, if $B \in \mathscr{B}$ with $T^{-1}B = B$ implies $m(B) = 1$ or $m(B) = 0$. We call m an ergodic measure.*

In other words, T is ergodic if and only if there is no non-trivial T-invariant set. The following are some equivalent conditions for ergodicity.

Theorem 3.5. *Let $(X, \mathscr{B}, m, T)$ be a MPS, then the following are equivalent.*
(i) T is ergodic.
(ii) Let $B \in \mathscr{B}$ with $m(T^{-1}B \triangle B) = 0$, then $m(B) = 1$ or 0.
(iii) Let $A \in \mathscr{B}$ with $m(A) > 0$, then $m(\bigcup_{i=1}^{\infty} T^{-i}A) = 1$.
(iv) Let $A, B \in \mathscr{B}$ with $m(A) > 0, m(B) > 0$, then there exists $n \in \mathbb{N}_+$ such that $m(T^{-n}A \cap B) > 0$.

Proof. (i)$\Rightarrow$(ii). Let $B \in \mathscr{B}$ with $m(T^{-1}B \triangle B) = 0$. We will construct a set $B_\infty \in \mathscr{B}$ with $T^{-1}B_\infty = B_\infty$ such that $m(B_\infty \triangle B) = 0$. Define

$$B_\infty = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} T^{-n}B,$$

it's clear $T^{-1}B_\infty = B_\infty$. Observe that

$$T^{-n}B \triangle B \subseteq \bigcup_{k=0}^{n-1} \left(T^{-(k+1)}B \triangle T^{-k}B\right) = \bigcup_{k=0}^{n-1} T^{-k}(T^{-1}B \triangle B),$$

hence $m(T^{-n}B \triangle B) = 0$ for any $n \in \mathbb{N}$. Since $(\bigcup_{n=k}^{\infty} T^{-n}B) \triangle B \subseteq \bigcup_{n=k}^{\infty}(T^{-n}B \triangle B)$, we have $m((\bigcup_{n=k}^{\infty} T^{-n}B) \triangle B) = 0$ for any $k \in \mathbb{N}$. Letting $k \to \infty$, we have $m(B_\infty \triangle B) = 0$ hence $m(B_\infty) = m(B)$. Since T is ergodic, $m(B_\infty) = 0$ or 1, therefore $m(B) = 0$ or 1.

(ii)$\Rightarrow$(iii). Let $A \in \mathscr{B}$ with $m(A) > 0$. Let $C = m(\bigcup_{i=1}^{\infty} T^{-i}A)$, then $T^{-1}C = m(\bigcup_{i=2}^{\infty} T^{-i}A) \subseteq C$. Since $m(T^{-1}C) = m(C)$, then $m(T^{-1}C \triangle C) = m(C \setminus T^{-1}C) = 0$. By (ii), $m(C) = 0$ or $m(C) = 1$, since $m(C) \geq m(T^{-1}A) = m(A) > 0$, we have $m(C) = 1$.

(iii)$\Rightarrow$(iv). Let $A, B \in \mathscr{B}$ with $m(A) > 0, m(B) > 0$. By (iii), $m(\bigcup_{n=1}^{\infty} T^{-n}A) = 1$, hence $m(\bigcup_{n=1}^{\infty}(T^{-n}A \cap B)) = m(B) > 0$, hence there exists $n \in \mathbb{N}_+$ such that $m(T^{-n}A \cap B) > 0$.

(iv)$\Rightarrow$(i). Let $B \in \mathscr{B}$ with $T^{-1}B = B$. Suppose $0 < m(B) < 1$, then both B and $X \setminus B$ have positive measure. By (iv), there is some n such that $m(T^{-n}B \cap (X \setminus B)) > 0$, but this is impossible since $T^{-n}B = B$ for any $n \in \mathbb{N}$. $\square$

There are also analogue equivalent conditions for ergodicity by using measurable functions.

Theorem 3.6. *Let $(X, \mathscr{B}, m, T)$ be a MPS, then the following are equivalent.*

(i) T is ergodic.

(ii) If f is measurable and $f(Tx) = f(x)$ for all $x \in X$, then f is constant a.e.

(iii) If f is measurable and $f(Tx) = f(x)$ a.e., then f is constant a.e.

(iv) If $f \in L^2(m)$ and $f(Tx) = f(x)$ for all $x \in X$, then f is constant a.e.

(v) If $f \in L^2(m)$ and $f(Tx) = f(x)$ a.e., then f is constant a.e.

Proof. For brevity, we prove (i)$\Leftrightarrow$(iii), the left can be proved in the same manner.

(iii)$\Rightarrow$(i). Let $B \in \mathscr{B}$ with $T^{-1}B = B$. Set $f = \chi_B$, then $f(Tx) = f(x)$ for all $x \in X$, hence by (iii), f is constant a.e., which implies $m(B) = 1$ or $m(B) = 0$.

(i)$\Rightarrow$(iii). Let f be measurable and $f(Tx) = f(x)$ a.e. By considering real and imaginary parts, we can assume f is real-valued. For $n \in \mathbb{N}_+$ and $j \in \mathbb{Z}$, define

$$A_{n,j} = \{x \in X : \frac{j}{n} \leq f(x) < \frac{j+1}{n}\},$$

then for each $n \in \mathbb{N}_+$, $X = \bigcup_{j=-\infty}^{\infty} A_{n,j}$ is a disjoint union. Since $T^{-1}A_{n,j} \bigtriangleup A_{n,j} \subseteq \{x : f(Tx) \neq f(x)\}$ and $f = f \circ T$ a.e., we have $m(T^{-1}A_{n,j} \bigtriangleup A_{n,j}) = 0$, then by ergodicity of T, $m(A_{n,j}) = 0$ or $m(A_{n,j}) = 1$. Hence for each n, there exists a unique $j = j_n$ such that $m(A_{n,j_n}) = 1$. Now let $B = \bigcap_{n=1}^{\infty} A_{n,j_n}$, then $m(B) = 1$. Since f can differ at most $\frac{1}{n}$ on each A_{n,j_n}, f is constant on B, hence f is constant a.e. $\square$

Now under the assumption that T is ergodic, we can reformulate the Birkhoff ergodic theory as follows.

Theorem 3.7 (Birkhoff Ergodic Theorem)**.** *Let $(X, \mathscr{B}, m)$ be a ergodic MPS and $f \in L^1(m)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f dm \ \text{a.e.} \tag{3.7}$$

Proof. By Theorem 3.2, we have $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} f(T^k x) = f^*(x)$ a.e., $f^*(Tx) = f(x)$ a.e. and $\int_X f^* dm = \int_X f dm$. Since T is ergodic, by Theorem 3.6 (iii) we have f^* is constant a.e., hence $\int_X f dm = f^*$ a.e. $\square$

In the above theorem, the left hand side and right hand side of (3.7) can be interpreted as the "time average" and the "space average" of f respectively. Birkhoff ergodic theorem shows that ergodicity is the right condition for "time average" and "space average" to be equal.

Now let us test the ergodicity of some examples we considered in Lecture 1.

Example 1. (Rotation on the circle).

Let $X = \mathbb{R} \setminus \mathbb{Z}$ and m be the Haar measure on X. For $\alpha \in (0,1)$, define $T : X \to X, Tx = x + \alpha (\bmod 1)$. Then T is ergodic if and only if α is irrational.

Proof. If $\alpha = \frac{p}{q} \in \mathbb{Q}$, define $f : X \to [0,1)$ by $f(x) = qx(\mathrm{mod}1)$, then $f(Tx) = f(x+\alpha) = q(x+\frac{p}{q})(\bmod 1) = f(x)$, but clearly f is not constant, hence T is not ergodic. For the converse implication, let $\alpha \notin \mathbb{Q}$. Let $g \in L^2(m)$ and $g(Tx) = g(x)$ for all x. Suppose g has Fourier expansion $g(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$, then $g(Tx) = g(x+\alpha) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \alpha} e^{2\pi i n x}$. Since $g(Tx) = g(x)$, by comparing the Fourier coefficients, we have $a_n = a_n e^{2\pi i n \alpha}$ for all $n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Q}$, we have $a_n = 0$ for all $n \neq 0$, hence $g = a_0$ a.e. $\square$

Example 2. (Doubling map on the circle).

Let $X = \mathbb{R} \setminus \mathbb{Z}$ and m be the Haar measure on X. Let $Tx = 2x(\mathrm{mod}1)$. Then T is ergodic.

Proof. Let $f \in L^2(m)$ and $f(Tx) = f(x)$ for all x. Let $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ be the Fourier series of f, then $f(Tx) = f(2x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i (2n) x}$. Since $f(Tx) = f(x)$, by comparing the Fourier coefficients, we have $a_n = a_{2n}$ for all $n \in \mathbb{Z}$ and $a_n = 0$ for any n odd, therefore $a_n = 0$ for all $n \neq 0$, hence $f = a_0$ a.e. $\square$

Example 1 can be generalized to rotation on compact group as follows.

Example 3. (Rotation on compact group).

Let K be a compact group. Let $a \in K$. Let m be the normalized Haar measure on K. Define $T : K \to K$ by $Tx = ax$. Then T is ergodic if and only if $\{a^n : n \in \mathbb{Z}\}$ is dense in K.

Proof. "$\Rightarrow$". Notice that T is a homeomorphism and preserves m. Let e be the identity of K. Let $O_e = \overline{\{a^n : n \in \mathbb{Z}\}} = \overline{\{T^n e : n \in \mathbb{Z}\}}$. Clearly $T^{-1} O_e = O_e$. Since T is ergodic, we have $m(O_e) = 0$ or $m(O_e) = 1$. Since $K = \bigcup_{b \in K} bO_e$ and $m(bO_e) = m(O_e)$ for any $b \in K$ by translation invariant of Haar measure, we have $m(O_e) = 1$. We claim $O_e = K$. Otherwise $K \setminus O_e$ is open hence has positive measure, which is a contradiction.

"$\Leftarrow$". This part needs an application of Fourier analysis on compact groups. Assume $O_e = K$. Let $f \in L^2(m)$ and $f \circ T = f$. Let $\hat{K}$ be the collection of characters of K, that is every element γ in $\hat{K}$ is a continuous homomorphism of K into the unit circle S^1. Elements in $\hat{K}$ are mutually orthogonal in $L^2(m)$, moreover there are uniquely determined complex numbers a_γ such that $f(x) = \sum_{\gamma \in \hat{K}} a_\gamma \gamma(x)$, since only countable terms in the summation are non-zero, we can assume $f(x) = \sum_i a_i \gamma_i(x)$, then $f(T(x)) = f(ax) = \sum_i a_i \gamma_i(a) \gamma_i(x)$. Since $f(Tx) = f(x)$, by comparing coefficients, we have $\gamma_i(a) = 1$ whenever $a_i \neq 0$, since $\gamma_i(a^n) = \gamma_i(a)^n = 1$ and $O_e = K$, we have $\gamma_i \equiv 1$. By orthogonality of γ_i, only the constant term can be non-zero, hence f is constant a.e. T is ergodic by Theorem 3.6. $\square$

The following theorem gives one more equivalent condition for ergodicity.

Theorem 3.8. *Let $(X, \mathscr{B}, m, T)$ be a MPS. Then T is ergodic if and only if ,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}A \cap B) = m(A)m(B), \tag{3.8}$$

for any $A, B \in \mathscr{B}$.

Proof. "$\Leftarrow$". Let $B \in \mathscr{B}$ with $T^{-1}B = B$. Put $A = B$ in (3.8), we get $m(B) = m(B)^2$, hence $m(B) = 0$ or $m(B) = 1$, hence T is ergodic.

"$\Rightarrow$". Let $A, B \in \mathscr{B}$. Set $f = \chi_A$ and apply Birkhoff ergodic theorem to f, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k x) = \int_X \chi_A dm = m(A) \text{ a.e.},$$

multiplying both sides by χ_B, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k x)\chi_B = \int_X \chi_A dm = m(A)\chi_B \text{ a.e.},$$

doing integral on both sides then applying the dominated convergence theorem, we obtain (3.8). $\square$

Usually to check ergodicity, we only need to consider a subset of $\mathscr{B}$ that generates $\mathscr{B}$.

Definition 3.3. *A collection $\mathscr{G}$ of subsets of X is called a semi-algebra if it satisfies*

(i) $\emptyset \in \mathscr{G}$.

(ii) If $A, B \in \mathscr{G}$, then $A \cap B \in \mathscr{G}$.

(iii) If $A \in \mathscr{G}$, then there exists a finite collection of sets $E_1, E_2, \cdots, E_k$ such that $X \setminus A = \bigcup_{i=1}^k E_i$ is a disjoint union and $E_i \in \mathscr{G}$ for each i.

A collection $\mathscr{A}$ of subsets of X is called an algebra if it satisfies (i), (ii) and (iii)$'$ $X \setminus A \in \mathscr{B}$ whenever $A \in \mathscr{B}$.

For example, let $X = [0, 1]$. Let $\mathscr{G}_1 = \{\emptyset\} \cup \{[0, a], (b, c] : 0 < a \leq 1, 0 < b < c \leq 1\}$, then $\mathscr{G}_1$ is a semi-algebra. Let $\mathscr{G}_2 = \{\emptyset\} \cup \{\text{all subintervals of } [0, 1]\}$, then $\mathscr{G}_2$ is a semi-algebra and it generates $\mathscr{B}(X)$, the Borel σ-algebra of X, since $\mathscr{B}(X) \supseteq \mathscr{B}(\mathscr{G}_2) \supseteq \{\text{all open subsets of } X\}$ and $\mathscr{B}(X) = \mathscr{B}(\{\text{all open subsets of } X\})$.

We denote the algebra generated by a semi-algebra $\mathscr{G}$ by $\mathscr{A}(\mathscr{G})$. Then it's easy to see $\mathscr{A}(\mathscr{G}) = \{\bigcup_{i=1}^n A_i : A_i \in \mathscr{G} \text{ are disjoint}, n \in \mathbb{N}\}$. Since we always assume $(X, \mathscr{B}, m)$ is complete, if $\mathscr{B}$ is generated by an algebra $\mathscr{A}$, then for any $A \in \mathscr{B}$,

$$m(A) = \inf\{\sum_{i=1}^{\infty} m(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i, A_i \in \mathscr{A} \text{ for all } i\},$$

see Halmos's "Measure Theory" for a proof. Using this, we can easily prove the following approximation lemma.

Lemma 3.9. *Let $(X, \mathscr{B}, m)$ be a probability space. Suppose $\mathscr{B}$ is generated by an algebra $\mathscr{A}$. Then for each $\epsilon > 0$ and $A \in \mathscr{B}$, there exists $A_0 \in \mathscr{A}$ such that*

$$m(A \triangle A_0) < \epsilon.$$

Proof. Fix $\epsilon > 0$ and $A \in \mathscr{B}$. As we have mentioned,

$$m(A) = \inf\{\sum_{i=1}^{\infty} m(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i, A_i \in \mathscr{A} \text{ for all } i\}.$$

Then there exists a sequence $\{A_i\} \subset \mathscr{A}$, such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$ and $\sum\limits_{i=1}^{\infty} m(A_i) < m(A) + \frac{\epsilon}{2}$, hence $m(\bigcup\limits_{i=1}^{\infty} A_i) < m(A) + \frac{\epsilon}{2}$. Since $m(\bigcup\limits_{i=1}^{\infty} A_i) = \lim\limits_{n\to\infty} m(\bigcup\limits_{i=1}^{n} A_i)$, there exist N such that $m(\bigcup\limits_{i=1}^{\infty} A_i) < m(\bigcup\limits_{i=1}^{N} A_i) + \frac{\epsilon}{2}$. Let $A_0 = \bigcup\limits_{i=1}^{N} A_i$, then $A_0 \in \mathscr{A}$. Moreover $m(A \setminus A_0) \leq \sum\limits_{i=1}^{\infty} m(A_i) - m(A_0) < \frac{\epsilon}{2}$ and $m(A_0 \setminus A) \leq \sum\limits_{i=1}^{\infty} m(A_i) - m(A) < \frac{\epsilon}{2}$, hence $m(A \triangle A_0) < \epsilon$.

$\square$

Now we can prove a more handy version of Theorem 3.8.

Theorem 3.10. *Let $(X, \mathscr{B}, m, T)$ be a MPS. Suppose $\mathscr{B}$ is generated by a semi-algebra $\mathscr{G}$. Then T is ergodic if and only if ,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}A \cap B) = m(A)m(B), \tag{3.9}$$

for any $A, B \in \mathscr{G}$.

Proof. Assume (3.9) holds for all elements of $\mathscr{G}$, we prove it also holds for all elements of $\mathscr{B}$. Since each element in $\mathscr{A} = \mathscr{A}(\mathscr{G})$, the algebra generated by $\mathscr{G}$, is a union of finite disjoint elements of $\mathscr{G}$, it's easy to check that (3.9) holds for elements of $\mathscr{A}$. Now fix $A, B \in \mathscr{B}$ and $\epsilon > 0$. By Lemma 3.9, there exist $A_0, B_0 \in \mathscr{A}$ such that $m(A \triangle A_0) < \epsilon$ and $m(B \triangle B_0) < \epsilon$, hence $|m(A) - m(A_0)| < \epsilon$ and $|m(B) - m(B_0)| < \epsilon$. Notice that

$$(T^{-k}A \cap B) \triangle (T^{-k}A_0 \cap B_0) \subseteq (T^{-k}A \triangle T^{-k}A_0) \cup (B \triangle B_0),$$

we have $|m(T^{-k}A \cap B) - m(T^{-k}A_0 \cap B_0)| < 2\epsilon$ for any k. Therefore

$$
\begin{aligned}
\Big|\frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k}A \cap B) - m(A)m(B)\Big| &\le \Big|\frac{1}{n}\sum_{k=0}^{n-1}(m(T^{-k}A \cap B) - m(T^{-k}A_0 \cap B_0))\Big| \\
&+ \Big|\frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k}A_0 \cap B_0) - m(A_0)m(B_0)\Big| \\
&+ m(A_0)|m(B) - m(B_0)| + m(B)|m(A) - m(A_0)| \\
&\le \Big|\frac{1}{n}\sum_{k=0}^{n-1}(m(T^{-k}A_0 \cap B_0) - m(A_0)m(B_0)\Big| + 4\epsilon.
\end{aligned}
$$

Letting $n \to \infty$, since (3.9) holds for A_0, B_0, we obtain

$$
\varlimsup_{n\to\infty} \Big|\frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k}A \cap B) - m(A)m(B)\Big| \le 4\epsilon,
$$

since $\epsilon > 0$ is arbitrary, we have proved that (3.9) holds for $A, B \in \mathscr{B}$. This completes the proof. $\square$