Suggested Solution to Assignment 7

Exercise 7.1

1. Suppose there exists one non-constant harmonic function u in D, which attains its maximum M at $\mathbf{x}_0 \in D$. Then by the mean value property, we have

$$u(\mathbf{x}_0) = \frac{1}{|B(\mathbf{x}_0, r)|} \iint_{B(\mathbf{x}_0,r)} u dS,$$

for all $\overline{B(\mathbf{x}_0, r)} \subset D$. Thus $u(\mathbf{x}) = M$ for all $\mathbf{x} \in \overline{B(\mathbf{x}_0, r)} \subset D$ since u is continuous and M is its maximum.

Now since u is not a constant, there exists $\mathbf{x}_1 \in D$ such that $u(\mathbf{x}_1) \neq u(\mathbf{x}_0)$. Since D is a region, we can find a continuous a curve $\gamma(t) \subset D$ $(0 \leq t \leq 1)$ such that $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}_1$. Set $E := \{0 \leq t \leq 1 | u(\gamma(t)) = M\}$, then E is closed since u and γ are both continuous. Let $t_0 \in E$, then by the above result, $u(\mathbf{x}) = M$ for all $\mathbf{x} \in \overline{B(\gamma(t_0), r)} \subset D$. So E is open. Hence, $E = [0, 1]$ since $E \neq \emptyset$ and $[0, 1]$ is connected. But this contradicts to $u(\mathbf{x}_1) \neq u(\mathbf{x}_0)$ and then we complete the proof. $\square$

2. Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$\Delta u = 0 \text{ in } D, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D.$$

Thus, by the Green's first identity for $v = u$, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n}\, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u\Delta u d\mathbf{x}.$$

Hence,

$$\iiint_D |\nabla u|^2 d\mathbf{x} = 0.$$

Therefore, $\nabla u \equiv 0$ and then $u(x)$ is a constant. $\square$.

3. Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$\Delta u = 0 \text{ in } D, \qquad \frac{\partial u}{\partial n} + au = 0 \text{ on } \partial D.$$

Thus, by the Green's first identity for $v = u$, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n}\, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u\Delta u d\mathbf{x}.$$

Hence,

$$\iiint_D |\nabla u|^2 d\mathbf{x} = -\iint_{\partial D} au^2 dS.$$

Since $a(\mathbf{x}) > 0$, $\nabla u \equiv 0$ and then $u(x)$ is a constant C. So by the Robin boundary conditions, we have $Ca(\mathbf{x}) = 0$. This shows that $C = 0$ and $u \equiv 0$. $\square$.

4. Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ satisfies

$$u_t - k\Delta u = 0 \text{ in } D \times (0, \infty),\ u = 0 \text{ on } \partial D \times (0, \infty),\ u(x, 0) = 0 \text{ in } D.$$

Thus, by the Green's first identity for $v = u$, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n}\, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u\Delta u d\mathbf{x}.$$

Therefore, by the diffusion equation

$$\begin{aligned} 0 &= \iiint_D |\nabla u|^2 d\mathbf{x} + \frac{1}{k}\iiint_D u u_t d\mathbf{x} \\ &= \iiint_D |\nabla u|^2 d\mathbf{x} + \frac{1}{2k}\frac{d}{dt}\iiint_D u^2 d\mathbf{x}. \end{aligned}$$

Define $E(t) := \iiint_D u^2 d\mathbf{x}$, then the above equality implies $\frac{d}{dt}E(t) \le 0$. Note that $E(0) = 0$ and $E(t) \ge 0$, we have $E(t) \equiv 0$. So we obtain $u \equiv 0$ and the uniqueness for the diffusion equation with Dirichlet boundary conditions is proved. □.

5. Suppose $u(\mathbf{x})$ minimizes the energy. Let $v(\mathbf{x})$ be any function and ϵ be any constant, then

$$E[u] \le E[u+\epsilon v] = E[u] + \epsilon \iiint_D \nabla u \cdot \nabla v d\mathbf{x} - \epsilon \iint_{\partial D} hv dS + \epsilon^2 \iiint_D |\nabla v|^2 d\mathbf{x}.$$

Hence, by calculus

$$\iiint_D \nabla u \cdot \nabla v d\mathbf{x} - \iint_{\partial D} hv dS = 0$$

for any function v. By Green's first identity,

$$-\iiint_D \nabla u \cdot v d\mathbf{x} + \iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} hv dS = 0.$$

Noting v is an arbitrary function, let v vanish on ∂D firstly and then the above equality implies $-\Delta u = 0$ in D. So the above equality changes to

$$\iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} hv dS = 0.$$

Since v is arbitrary, $\dfrac{\partial u}{\partial n} = h$ on ∂D and then $u(\mathbf{x})$ is a solution of the following Neumann problem

$$-\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = h \text{ on bdy } D.$$

Note that the Neumann problem has a unique solution up to constant and the functional $E[w]$ does not change if a constatn is added to w, thus it is the only funtion that can minimize the energy up to a constant.

Note that here we assume functions u, v and the domain D are smooth enough, at least under which the Green's first identity can be hold. □

6. (a) Suppose u_1 and u_2 are solutions of the problem. Then $u = u_1 - u_2$ tends to zero at infinity, and is constant on ∂A and constant on ∂B, and satisfies

$$\iint_{\partial A} \frac{\partial u}{\partial n} dS = 0 = \iint_{\partial B} \frac{\partial u}{\partial n} dS.$$

Suppose that $u \ne 0$, then without loss of generality, we assume that there exists $\mathbf{x}_0 \in \bar{D}$ such that $u(\mathbf{x}_0) > 0$. Since u tends to zero at infinity, so there exists $R \gg 1$ such that $u(\mathbf{x}) \le \frac{u(\mathbf{x}_0)}{2}$ if $|\mathbf{x}| \ge R$. Then u is a harmonic function in $D \cap B(\mathbf{0}, R)$ and $\max\limits_{\partial B(\mathbf{0},R)} u \le \frac{u(\mathbf{x}_0)}{2} \le u(\mathbf{x}_0) \le \max\limits_{D\cap B(\mathbf{0},R)} u$. Thus, the maximum is attained on ∂A or ∂B by the Maximum Principle. WOLG, we assume that u attain its maximum on ∂A and then by any point of ∂A is a maximum point since u is constant on ∂A. Hence, by the Hopf maximum principle, $\partial u/\partial n > 0$ on ∂A, but this contradicts to the condition

$$\iint_{\partial A} \frac{\partial u}{\partial n} dS = 0.$$

Therefore, $u \equiv 0$ and then the solution is unique.

(b) If not, then $u(\mathbf{x})$ have a negative minimum. As above, by the Minimum Principle, we get that u attains its minimum on ∂A or ∂B. WOLG, we assume that u attains its minimum on ∂A, then $\frac{\partial u}{\partial n} < 0$ by the Hopf principle since u is not a constant. But this contradicts to $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$. So $u \geq 0$ in D,

(c) Suppose that there exists $\mathbf{x}_0 \in D$ such that $u(\mathbf{x}_0) = 0$. Then by (b) and the Strong Minimum Principle, u is a constant in D. But this contradicts to $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$. So $u > 0$ in D. (Actually , we can prove that $u > 0$ on ∂A and ∂B by the Hopf principle again as in (b)). ☐

7. (Extra Problem 1) Suppose u_1 and u_2 are solutions of the problem. Then

$$\triangle(u_1 - u_2) = u_1^3 - u_2^3 = (u_1 - u_2)(u_1^2 + u_1 u_2 + u_2^2) \ in \ D$$

$$\frac{\partial(u_1 - u_2)}{\partial n} + a(x)(u_1 - u_2) = 0 \ on \ \partial D$$

Thus,

$$\begin{aligned} 0 \leq \int_D (u_1 - u_2)^2(u_1^2 + u_1 u_2 + u_2^2) &= \int_D (u_1 - u_2)\triangle(u_1 - u_2) \\ &= \int_{\partial D} (u_1 - u_2)\frac{\partial(u_1 - u_2)}{\partial n} - \int_D |Du_1 - Du_2|^2 \\ &= -\int_{\partial D} a(x)(u_1 - u_2)^2 - \int_D |Du_1 - Du_2|^2 \leq 0 \end{aligned}$$

since $a(x) \geq 0$. $\Rightarrow \int_D (u_1 - u_2)^2(u_1^2 + u_1 u_2 + u_2^2) = 0 \Rightarrow u_1 = u_2 \ in \ D$

8. (Extra Problem 2) (a)

$$E[u] := \int_D \frac{1}{2}(|Du|^2 + b(x)u^2) + f(x)u$$

(b)If u is a solution, then $\forall v \in C_0^2(D),\ 0 = \int_D -v\triangle u + b(x)uv + f(x)v = \int_D Du \cdot Dv + b(x)uv + f(x)v$ $\Rightarrow \forall w \in C^2(D),\ w = h \ on \ \partial D$, take $v = w - u$,

$$\begin{aligned} E[w] &= \int_D \frac{1}{2}|Dv|^2 + \frac{1}{2}|Du|^2 + Dv \cdot Du + \frac{1}{2}b(x)v^2 + \frac{1}{2}b(x)u^2 + b(x)uv + f(x)v + f(x)u \\ &= \int_D \frac{1}{2}|Dv|^2 + \frac{1}{2}b(x)v^2 + E[u] \\ &\geq E[u] \end{aligned}$$

since $b(x) \geq 0$.

Conversely,$\forall v \in C_0^2(D), \forall t \in \mathbb{R},\ w := u + tv \in C^2,\ w = u = h \ on \ \partial D$

$$\Rightarrow 0 = \frac{d}{dt}E[u + tv]\big|_{t=0} = \int_D Du \cdot Dv + b(x)uv + f(x)v$$

$$\Rightarrow \int_D v(-\triangle u + b(x)u + f(x)) = 0,\ \forall v \in C_0^2(D)$$

$$\Rightarrow -\triangle u + b(x)u + f(x) = 0 \ in \ D$$

Exercise 7.2

2. Let $r = |\mathbf{x}|$ and $D_\epsilon := \{x|\epsilon < r < R\}$, where R is large enough such that $\phi = 0$ outside $r < R/2$. Using the Green's second identity,

$$\iiint_{D_\epsilon} \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})d\mathbf{x} = \iint_{\partial D_\epsilon}\left[\frac{1}{|\mathbf{x}|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{\mathbf{x}}\right]dS$$
$$= \iint_{|\mathbf{x}|=R}\left[\frac{1}{|\mathbf{x}|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{\mathbf{x}}\right]dS + \iint_{|\mathbf{x}|=\epsilon}\left[\frac{1}{|\mathbf{x}|}\cdot\frac{\partial\phi}{\partial n} - \phi\cdot\frac{\partial}{\partial n}\frac{1}{\mathbf{x}}\right]dS$$
$$= -\iint_{|\mathbf{x}|=\epsilon}\left[\frac{1}{\epsilon}\cdot\frac{\partial\phi}{\partial r} + \phi\cdot\frac{1}{c^2}\right]dS = -4\pi\overline{\phi} - 4\pi\epsilon\overline{\frac{\partial\phi}{\partial r}},$$

where $\overline{\phi}$ denotes the average value of ϕ on the sphere $\{r = \epsilon\}$, and $\overline{\frac{\partial\phi}{\partial r}}$ denotes the average value of $\frac{\partial\phi}{\partial r}$ on this sphere. Since ϕ is continuous and $\frac{\partial\phi}{\partial r}$ is bounded,

$$-4\pi\overline{\phi} - 4\pi\epsilon\overline{\frac{\partial\phi}{\partial r}} \to -4\pi\phi(\mathbf{0}) \quad \text{as } \epsilon \to 0.$$

Hence,

$$\phi(\mathbf{0}) = -\iiint\limits_{|\mathbf{x}|<R} \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}. \qquad \square$$

3. Choosing $D = B(\mathbf{x}_0, R)$ in the representation formula (1) and using the divergence theorem,

$$u(\mathbf{x}_0) = \iint\limits_{\partial B(\mathbf{x}_0,R)}\left[-u(\mathbf{x})\cdot\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{1}{|\mathbf{x}-\mathbf{x_0}|}\cdot\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi}$$
$$= \iint\limits_{|\mathbf{x}-\mathbf{x}_0|=R}\left[\frac{1}{R^2}u(\mathbf{x}) + \frac{1}{R}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi}$$
$$= \frac{1}{4\pi R^2}\iint\limits_{|\mathbf{x}-\mathbf{x}_0|=R} udS + \frac{1}{4\pi R}\iiint\limits_{|\mathbf{x}-\mathbf{x}_0|<R}\Delta u d\mathbf{x}$$
$$= \frac{1}{4\pi R^2}\iint\limits_{|\mathbf{x}-\mathbf{x}_0|=R} udS. \qquad \square$$

4. (Extra Problem) $\forall\phi(x) \in C_c^2(\mathbb{R}^2)$, we need to show that

$$\phi(0) = \int\int_{\mathbb{R}^2} \log|x|\triangle\phi(x)\frac{dx}{2\pi}$$

proof: Let $r = |x|$ and $D_\varepsilon := x : \varepsilon < r < R$, where R is large enough such that $\phi = 0$ if $r \geq R/2$. Using the Green's identity,

$$\int\int_{D_\varepsilon} \log|x|\Delta\phi(x)dx = \int_{\partial D_\varepsilon}[\log|x|\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial}\log|x|]dS$$
$$= \int_{|x|=\varepsilon}[\log|x|\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial}\log|x|]dS$$
$$= -\int_{|x|=\varepsilon}[\log\varepsilon\frac{\partial\phi}{\partial r} - \phi\frac{1}{\varepsilon}]dS$$
$$= 2\pi\bar{\phi} - 2\pi\varepsilon\log\varepsilon\frac{\bar{\partial}\phi}{\partial r}$$

where $\bar{\phi}$, $\bar{\frac{\partial \phi}{\partial r}}$ denote the average value of ϕ, $\frac{\partial \phi}{\partial r}$ on the sphere $|x| = \varepsilon$. Since ϕ is continuous and $\frac{\partial \phi}{\partial r}$ is bounded,

$$2\pi\bar{\phi} - 2\pi\varepsilon\log\varepsilon\bar{\frac{\partial \phi}{\partial r}} \to 2\pi\phi(0), \ as \ \varepsilon \to 0$$

Exercise 7.3

1. Suppose G_1 and G_2 both are the Green's functions for the operator $-\Delta$ and the domain D at the point $\mathbf{x}_0 \in d$. By (i) and (iii), we obtain that $G_1(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$ and $G_2(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$ both are harmonic functions in D. Thus, by (ii) and the uniqueness theorem of harmonic function, we have

$$G_1(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|} = G_2(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|},$$

that is, the Green's function is unique. □

3. In the textbook,

$$A_\epsilon = \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS,$$

where $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$ and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$. Note that

$$u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x}, \mathbf{a})$$

, where $H(\mathbf{x}, \mathbf{a})$ is a harmonic function throughout the domain D. What's more, $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$ is a harmonic function in $\{|\mathbf{x}-\mathbf{a}| < \epsilon\}$, then we have

$$\begin{aligned}
A_\epsilon &= \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x}, \mathbf{a}))\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}(-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|} + H(\mathbf{x}, \mathbf{a}))\right] dS \\
&= \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[-\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|}\frac{\partial v}{\partial n} + v\frac{\partial}{\partial n}(\frac{1}{4\pi|\mathbf{x}-\mathbf{a}|})\right] dS \\
&\quad + \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[H(\mathbf{x}, \mathbf{a})\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}H(\mathbf{x}, \mathbf{a})\right] dS \\
&= v(\mathbf{a}) - \iiint_{|\mathbf{x}-\mathbf{a}|<\epsilon} [H(\mathbf{x}, \mathbf{a})\Delta v - v\Delta H(\mathbf{x}, \mathbf{a})]\, d\mathbf{x} = v(\mathbf{a}).
\end{aligned}$$

Here we use the representation formula (7.2.1) and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. So $\lim_{\epsilon \to 0} A_\epsilon = v(\mathbf{a}) = G(\mathbf{a}, \mathbf{b})$. □

Exercise 7.4

1. By (i), we know that $G(x)$ is a linear function in $[0, x_0]$ and $[x_0, l]$. Since (ii) and $G(x)$ is continuous at x_0, we have

$$G(x) = \begin{cases} kx, & 0 < x \le x_0; \\ \frac{kx_0}{x_0 - l}(x - l), & x_0 < x < l. \end{cases}$$

Hence, $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0 if and only if

$$(G(x) + \frac{1}{2}|x-x_0|)'\Big|_{x_0^-} = (G(x) + \frac{1}{2}|x-x_0|)'\Big|_{x_0^+},$$

if and only if

$$k = \frac{l - x_0}{l}.$$

So the one-dimensional Green's function for the interval $(0,l)$ at the point $x_0 \in (0,l)$ is

$$G(x) = \begin{cases} \frac{l-x_0}{l}x, & 0 < x \le x_0; \\ -\frac{x_0}{l}(x-l), & x_0 < x < l. \end{cases} \qquad \square$$

2. Assume that $h(x,y)$ is a continuous function that vanished outside $\{(x,y)|x^2+y^2 \le R^2\}$ and $|h(x,y)| \le M$. Then by the formula (3), we have

$$\begin{aligned} |u(x_0,y_0,z_0)| &\le \frac{M}{2\pi} \iint\limits_{\{(x,y)|x^2+y^2\le R^2\}} [(x-x_0)^2+(y-y_0)^2+z_0^2]^{-\frac{3}{2}} dxdy \\ &\le \frac{MR^2}{2(\sqrt{x_0^2+y_0^2+z_0^2}-R)^3}, \quad \text{when } \sqrt{x_0^2+y_0^2+z_0^2} > R. \end{aligned}$$

Therefore, u satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0, \qquad \text{as } |\mathbf{x}| \to \infty. \qquad \square$$

3. From (3), we have

$$\begin{aligned} u(x_0,y_0,z_0) &= \frac{z_0}{2\pi} \iint [(x-x_0)^2+(y-y_0)^2+z_0^2]^{-\frac{3}{2}} h(x,y) dxdy \\ &= \frac{z_0}{2\pi} \iint [x^2+y^2+z_0^2]^{-\frac{3}{2}} h(x+x_0,y+y_0) dxdy. \end{aligned}$$

Now we change variables such that $x = z_0 s\cos\theta$, $y = z_0 s \sin\theta$. Then

$$\begin{aligned} u(x_0,y_0,z_0) &= \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^\infty (z_0^2 s^2 + z_0^2)^{-\frac{3}{2}} h(z_0 s\cos\theta + x_0, z_0 s\sin\theta + y_0) z_0^2 s \; dsd\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty s(s^2+1)^{-\frac{3}{2}} h(z_0 s\cos\theta + x_0, z_0 s \sin\theta + y_0) \; dsd\theta. \end{aligned}$$

Since

$$\int_0^\infty s(s^2+1)^{-\frac{3}{2}} ds = -(s^2+1)^{-\frac{3}{2}}\Big|_0^\infty = 1,$$

$$\lim_{z_0\to 0} u(x_0,y_0,z_0) = h(x_0,y_0),$$

if the limit can be taken in the integration, for example, when $h(x,y)$ is bounded. $\square$

5. Here is **one of suggested explains**.
Since the half-plane $\{y > 0\}$ is not bounded, this only means that the solution is not unique under the (uncompleted) boundary condition or it is not a well-posed problem. But generally it will be a well-posed problem if we add another boundary condition, such as

$$u(\mathbf{x}) \to 0, \qquad \text{as } |\mathbf{x}| \to \infty. \qquad \square$$

6. (a) Using the method of reflection, as in the dimension three, we have

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| - \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0^*|,$$

where $\mathbf{x} = (x,y)$, $\mathbf{x}_0 = (x_0,y_0)$, $\mathbf{x}_0^* = (x_0,-y_0)$.

(b) Since

$$-\frac{\partial G}{\partial y} = \frac{y+y_0}{2\pi[(x-x_0)^2+(y+y_0)^2]} - \frac{y-y_0}{2\pi[(x-x_0)^2+(y-y_0)^2]}$$
$$= \frac{y_0}{\pi[(x-x_0)^2+y_0^2]}$$

on $y=0$, the solution is given by

$$u(x_0,y_0) = \frac{y_0}{\pi}\int \frac{h(x)}{(x-x_0)^2+y_0^2}dx.$$

(c) By (b), we have

$$u(x_0,y_0) = \frac{y_0}{\pi}\int \frac{h(x)}{(x-x_0)^2+y_0^2}dx = \frac{1}{\pi}\int \frac{1}{x^2+1}dx = 1. \qquad \square$$

7. (a) If $u(x,y) = f(\frac{x}{y})$, then

$$u_x = \frac{1}{y}f'(\frac{x}{y}),\ u_y = -\frac{x}{y^2}f'(\frac{x}{y}).$$
$$u_{xx}+u_{yy} = \frac{1}{y^2}f''(\frac{x}{y}) + \frac{x^2}{y^4}f''(\frac{x}{y}) + \frac{2x}{y^3}f'(\frac{x}{y}).$$

So f has to satisfy the following ODE

$$(1+t^2)f''(t)+2tf'(t)=0.$$

$$f(t) = \int_0^t \frac{a}{1+s^2}ds + b = a\arctan t + b,$$

where a, b are contants.

(b)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{y}f'(\frac{x}{y})\frac{x}{r} - \frac{x}{y^2}f'(\frac{x}{y})\frac{y}{r} = 0.$$

(c) Using the polar coordinate,

$$v(r,\theta) = c\theta + d,$$

where c, d are constants. So in the (x,y)-coordinate, we have

$$v(x,y) = c\arctan\frac{y}{x} + d.$$

(d) By (a),

$$u(x,y) = f(\frac{x}{y}) = a\arctan\frac{x}{y} + b.$$

So

$$h(x) = \lim_{y\to 0} u(x,y) = \begin{cases} \frac{\pi}{2}a+b, & x>0; \\ b, & x=0; \\ -\frac{\pi}{2}a+b, & x<0. \end{cases}$$

(e) By (d), we see that $h(x)$ is not continuous unless u is a constant. This agrees with the condition that $h(x)$ is continuous in Exercise 6. $\square$

9. Here the methof of reflection is used. Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ such that $ax_0 + by_0 + cz_0 > 0$, then its reflection point $\mathbf{x}_0' = (x_0', y_0', z_0')$ about the hyperplane $\{ax + by + cz = 0\}$ satisfies

$$x_0' = x_0 - \frac{2a}{a^2+b^2+c^2}(ax_0 + by_0 + cz_0),$$
$$y_0' = y_0 - \frac{2b}{a^2+b^2+c^2}(ax_0 + by_0 + cz_0),$$
$$z_0' = z_0 - \frac{2c}{a^2+b^2+c^2}(ax_0 + by_0 + cz_0).$$

So the Green's function for the tilted half-space $\{(x, y, z) : \ ax + by + cz > 0\}$ is given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0'|}. \qquad \square$$

10. In case $\mathbf{x}_0$, the formula for the Green's function is

$$G(\mathbf{x}, \mathbf{0}) = -\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a},$$

since $-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a} \in C^2$ and is harmonic in $B_a(\mathbf{0})$, except at the point $\mathbf{x} = \mathbf{0}$; $[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}]\Big|_{|\mathbf{x}|=a} = 0$; and $[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}] + \frac{1}{4\pi|\mathbf{x}|}$ is harmonic at $\mathbf{0}$. $\square$

13. Let $\mathbf{x}_0 \in D$, then we have

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi|r_0\mathbf{x}/a - a\mathbf{x}_0/r_0|}$$

is the Green's function at $\mathbf{x}_0$ for the whole ball, where a is the radius and $r_0 = |\mathbf{x}_0|$.

Let $\mathbf{x}_0' = (x_0, y_0, -z_0)$, then

$$G(\mathbf{x}, \mathbf{x}_0') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0'|} + \frac{1}{4\pi|r_0\mathbf{x}/a - a\mathbf{x}_0'/r_0|}$$

is the Green's function at $\mathbf{x}_0'$ for the whole ball. Note that $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}, \mathbf{x}_0')$ on ∂D and $G(\mathbf{x}, \mathbf{x}_0')$ is a harmonic function on D. Hence, $G(\mathbf{x}, \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}_0')$ is the Green's function for D by the uniqueness of the Green's function. $\square$

15. (a) If $v(x, y)$ is harmonic and $u(x, y) = v(x^2 - y^2, 2xy)$, then

$$u_x = 2xv_x + 2yv_y, \qquad u_y = -2yv_x + 2xv_y.$$

Thus

$$\begin{aligned} u_{xx} + u_{yy} &= 2v_x + 4x^2 v_{xx} + 4xyv_{xy} + 4xyv_{yx} + 4y^2 v_{yy} \\ &\quad - 2v_x + 4y^2 v_{xx} - 4xyv_{xy} - 4xyv_{yx} + 4x^2 v_{yy} \\ &= 4(x^2 + y^2)(v_{xx} + v_{yy}) = 0. \end{aligned}$$

(b) Using the polar coordinates, the transformation si

$$(r\cos\theta, r\sin\theta) \mapsto (r^2\cos 2\theta, r^2 \sin 2\theta).$$

Therefore, it maps the first quadrant onto the half-plane $\{y > 0\}$. Note that the transformation is one-one which also maps the x-positive-axis to x-positive-axis and y-positive-axis to x-negative-axis. $\square$

17. (a) Here the method of reflection is used and you can also use the result of Exercise 155 to find the Green's function for the quadrant Q. Let $\mathbf{x}_0 = (x_0, y_0) \in Q$, $\mathbf{x}_0^y = (-x_0, y_0)$, $\mathbf{x}_0^x = (x_0, -y_0)$ and $\mathbf{x}_0^0 = (-x_0, -y_0)$. Then it is easy to verify that

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| - \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0^y| - \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0^x| + \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0^0|$$

is the Green's function at $\mathbf{x}_0$ for the quadrant Q.

(b) (Extra Problem 3)

$$\frac{\partial G}{\partial n}\Big|_{x=0,y>0} = -\frac{\partial G}{\partial x} = \frac{x_0}{\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0^0|^2}\right)$$
$$\frac{\partial G}{\partial n}\Big|_{x>0,y=0} = -\frac{\partial G}{\partial y} = \frac{y_0}{\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0^0|^2}\right)$$

By formula (1) in Section 7.3 in the textbook,

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n}dS$$
$$= \frac{x_0}{\pi}\int_0^\infty g(y)\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0^0|^2}\right)dy + \frac{y_0}{\pi}\int_0^\infty h(x)\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0^0|^2}\right)dx \qquad \square$$

19. Consider the four-dimensional laplacian $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$. Show that its fundamental solution is r^{-2}, where $r^2 = x^2 + y^2 + z^2 + w^2$. proof: Direct computation according to its definition. $\square$

20. Using the conclusion in 17 and 20, we have the singular part of the Green function is $-\frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0|^{-2}$, where ω_4 is the volume of S^3, $\mathbf{x} = (x, y, z, w)$ and $\mathbf{x}_0 = (x_0, y_0, z_0, w_0)$. So the Green function is

$$-\frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0|^{-2} + \frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0^-|^{-2},$$

where $\mathbf{x}_0^- = (x_0, y_0, z_0, -w_0)$. $\square$

21. By the formula (2)(3) in Section7.3, we get

$$u(\mathbf{x}_0) = \int\int_{bdyD}\left(u\frac{\partial N}{\partial n} - \frac{\partial u}{\partial n}N\right) = \int\int_{bdyD} -\frac{\partial u}{\partial n}N$$

since $\frac{\partial N}{\partial n} = 0$ on the boundary. Thus we proved the following theorem:

If $N(\mathbf{x}, \mathbf{x}_0)$ is the Neumann function, then the solution of the Neumann problem is given by the formula

$$u(\mathbf{x}_0) = -\int\int_{bdyD}\frac{\partial u}{\partial n}N$$

22. (Extra Problem 4)By the formula (4) in Section7.4,

$$u(\mathbf{x}_0) = \frac{z_0}{2\pi}\int\int_{z=0}\frac{\frac{1}{\sqrt{x^2+y^2+1}}}{|\mathbf{x}-\mathbf{x}_0|^3}dS.$$

If the bounded condition is dropped, then we have $v = u + az$ where a is arbitrary real number is also a solution.