Suggested Solution to Homework 7

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P207, 4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \qquad \text{and} \qquad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2, \qquad T^* = T_1 - iT_2.$$

Show uniqueness, that is $T_1 + iT_2 = S_1 + iS_2$ implies $S_1 = T_1$ and $S_2 = T_2$; here S_1 and S_2 are self-adjoint by assumption.

Proof. Let T be a bounded linear operator on H. Set $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$. Then,

$$T_1^* = (\frac{1}{2}(T + T^*))^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = T_1$$
$$T_1^* = (\frac{1}{2i}(T + T^*))^* = -\frac{1}{2i}(T^* - (T^*)^*) = \frac{1}{2i}(-T^* + T) = T_2$$

So, T_1 and T_2 are self-adjoint. Moreover, it is easy to check that

$$T = T_1 + iT_2, \qquad T^* = T_1 - iT_2.$$

Assume $T_1 + iT_2 = S_1 + iS_2$ with S_1, S_2 are self-adjoint. Then,

$$(T_1 + iT_2)^* = (S_1 + iS_2)^* \Leftrightarrow T_1 - iT_2 = S_1 - iS_2.$$

Therefore, $T_1 = S_1$ and $T_2 = S_2$.

□

P374, 3.(Invariant subspace) A subspace Y of a normed space X is said to be invariant under a linear operator $T : X \to X$ if $T(Y) \subset Y$. Show that an eigenspace of T is invariant under T. Give examples.

Proof. Let Y be an eigenspace of T corresponding to the eigenvalue λ. Then $Tx = \lambda x, \forall x \in Y$. Thus,

$$T(Tx) = T(\lambda x) = \lambda Tx.$$

So, $Tx \in Y$, i.e. Y is invariant.

Example: Consider $X = \mathbb{R}^2$ and $T(x, y) = (x, 0), \forall (x, y) \in \mathbb{R}^2$. Then $Y = \{(x, 0)|x \in \mathbb{R}\}$ is an eigenspace of T corresponding to the eigenvalue 1. Obviously, $T(x, 0) = (x, 0) \in Y$. So, Y is invariant.

□

P374, 5. Let (e_k) be a total orthonormal sequence in a separable Hilbert space H and let $T : H \to H$ be defined at e_k by

$$Te_k = e_{k+1}, \qquad (k = 1, 2, \cdots)$$

and then linearly and continuously extended to H. Find invariant subspaces. Show that T has no eigenvalues.

Proof. Let $Y_n = \text{span}\{e_k\}_{k \geq n}$. Then Y_n are invariant subspaces. Indeed, $\forall x \in Y_n$, $x = \sum_{k=n}^{\infty} \alpha_k e_k$. Then

$$T(x) = T(\sum_{k=n}^{\infty} \alpha_k e_k) = \sum_{k=n}^{\infty} \alpha_k T(e_k) = \sum_{k=n}^{\infty} \alpha_k e_{k+1} \in Y_n.$$

Now we claim that T has no eigenvalue. Otherwise, suppose λ is a eigenvalue of T, i.e. $Tx = \lambda x$ for some $x \neq 0$. Since H is separable Hilbert space, and (e_k) is its total orthonormal sequence. Then $x = \sum_{k=1}^{\infty} \alpha_k e_k$ so that $Tx = \sum_{k=1}^{\infty} \alpha_k e_{k+1} = \lambda \sum_{k=1}^{\infty} \alpha_k e_k$. Thus $\lambda\alpha_1 = 0$ and $\lambda\alpha_k = \alpha_{k-1}$. Hence $\alpha_k = 0, \forall k \in \mathbb{N}$. A contradiction!

□

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