MATH 2040A Linear Algebra II 2015-16

Week 3 : Eigenvalues / Eigenvectors, Diagonalizability

(textbook §5.1 and 5.2)

Eigenvalues / Eigenvectors for $T: V \to V$

Given linear $T: V \to V$, $\dim V = n < \infty$ and an ordered basis β, we have the following commutative diagram:

$$\begin{tikzcd} v \in V \arrow[r, "T"] \arrow[d, "\cong_\beta"'] & V \arrow[d, "\cong_\beta"] \\ \mathbb{F}^n \arrow[r, "A = [T]_\beta"'] & \mathbb{F}^n \end{tikzcd}$$

Prop: $v \in V$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{F}$

$\Leftrightarrow$ $[v]_\beta$ is an eigenvector of $A = [T]_\beta$ with eigenvalue $\lambda \in \mathbb{F}$

Proof: The above diagram "commutes" means that

$$\forall v \in V, \qquad A[v]_\beta = [Tv]_\beta$$

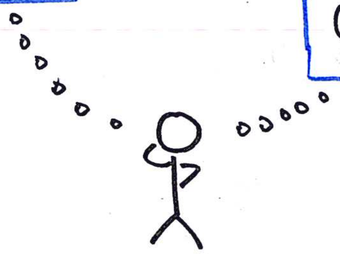
Hence, if v is an eigenvector of T, i.e. $Tv = \lambda v$ $\quad (v \neq \vec{0})$

$$\Rightarrow \quad A[v]_\beta = [Tv]_\beta = [\lambda v]_\beta \overset{\cong_\beta \text{ is linear}}{=} \lambda [v]_\beta$$

i.e. $[v]_\beta$ is an eigenvector of A with the same eigenvalue λ.

By reversing the argument, we proved the proposition. $\square$

Finding eigenvalues / eigenvectors of T	reduces to $\rightsquigarrow$	Finding eigenvalues / eigenvectors of $A = [T]_\beta$ (for ANY basis β)



Prop.: Let $A, B \in M_{n\times n}(\mathbb{F})$ be similar matrices, i.e.

$\exists$ invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t. $B = Q^{-1}AQ$. Then

(i) The eigenvalues for A and B are the same (even with multiplicity).

(ii) $v \in \mathbb{F}^n$ is an eigenvector of B $\Leftrightarrow$ $Qv \in \mathbb{F}^n$ is an eigenvector of A (with the same eigenvalue.)

Proof: (i) char. poly. of $B :=$
$$\det(B-\lambda I)$$
$$= \det(Q^{-1}AQ - \lambda I) \qquad (B = Q^{-1}AQ)$$
$$= \det\left(Q^{-1}(A-\lambda I)Q\right)$$
$$= (\det Q)^{-1} \cdot \det(A-\lambda I)\cdot \det Q \qquad (\det(AB) = \det A \cdot \det B)$$
$$= \det(A-\lambda I)$$
$$=: \text{char. poly. of } A$$

Same char. poly. $\Rightarrow$ Same eigenvalues (with multiplicity).

(ii)
$$Bv = \lambda v \Leftrightarrow Q^{-1}AQv = \lambda v$$
$$\Leftrightarrow AQv = Q(\lambda v)$$
$$\Leftrightarrow A(Qv) = \lambda(Qv)$$

$\square$

Similar matrices come from the same linear transformation, just with different basis. Therefore, similar matrices should have a lot of "properties" in common!

Diagonalizability

Recall the fundamental question:

Q: Given $T : V \to V$ linear, can we "diagonalize" it?

i.e. $\exists$ basis β for V s.t. $[T]_\beta$ is diagonal?

Equivalently, given $A \in M_{n\times n}(\mathbb{F})$, $\exists$ invertible $Q \in M_{n\times n}(\mathbb{F})$

s.t. $Q^{-1}AQ$ is diagonal?

Def$^{\underline{n}}$: An eigenbasis of V (or $\mathbb{F}^n$) for $T : V \to V$ (or $A \in M_{n\times n}(\mathbb{F})$)

is a basis of V (or $\mathbb{F}^n$) consisting of eigenvectors.

Note:
- eigenbasis exists $\iff$ T (or A) is diagonalizable.
- eigenbasis (if exists) is NOT unique.

E.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ ANY basis is an eigenbasis !!

(Q: why?)

(Exercise: What other matrices have this property?)

Def$^{\underline{n}}$: If $\lambda \in \mathbb{F}$ is an eigenvalue of T (or A), then the eigenspace of T (or A) corresponding to λ is the subspace

$$\boxed{E_\lambda := N(T - \lambda I)} \qquad \left(\text{or } E_\lambda := N(A - \lambda I)\right)$$

Note:
- $E_\lambda = \left\{ \begin{matrix} \text{eigenvectors with} \\ \text{eigenvalue } \lambda \end{matrix} \right\} \cup \{\vec{0}\}$.
- $E_\lambda \neq \{\vec{0}\}$, i.e. $\dim E_\lambda \geq 1$.

(because $\exists\ \vec{0} \neq v \in E_\lambda$ if λ is an eigenvalue.)

eigenvector.

Recall the "flow chart" of finding eigenbasis (if exists):

Given an $n \times n$ matrix A

↓

Compute its characteristic polynomial

$f(\lambda) = \det(A - \lambda I)$ $\quad$ $\deg f = n$

↓

Solve the characteristic equation $\quad$ (hard)

$f(\lambda) = 0$ $\quad$ deg n polynomial equation

get all the roots (in $\mathbb{F}$)

(distinct) eigenvalues: $\quad \lambda_1 \quad \lambda_2 \quad \cdots\cdots \quad \lambda_k$ $\quad$ $\boxed{k \leq n}$

find each eigenspace

eigenspaces: $\quad E_{\lambda_1} := N(A - \lambda_1 I) \quad E_{\lambda_2} := N(A - \lambda_2 I) \quad \cdots \quad E_{\lambda_k} := N(A - \lambda_k I)$

get a basis for each eigenspace

$\beta_1 \quad \beta_2 \quad \cdots\cdots \quad \beta_k$

put them together

eigenbasis: $\quad \beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$

(if exists)

If there are "enough" eigenvectors, then I can put them together to get an eigenbasis.

- Of course, since v is an eigenvector $\Rightarrow a\cdot v$ is an eigenvector for <u>any</u> $a \in \mathbb{F}$, there always exist infinitely many eigenvectors if there exists one. Therefore, the key point is how many <u>linearly independent</u> eigenvectors can we get! For an $n\times n$ matrix A we need n linearly independent eigenvectors.

- The following Theorem tells us that if we follow the flow chart before, linear independence of β is automatic!

<u>Theorem</u>: Let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be <u>distinct</u> eigenvalues of $T: V \to V$. If $\vec{0} \neq v_i \in E_{\lambda_i}$, $i = 1, \dots, k$, then $\{v_1, \dots, v_k\}$ is linearly independent.

<u>Proof</u>: By induction on k.

$\boxed{\text{Base case } k=1}$: Trivial, since $v_1 \neq \vec{0}$ (as eigenvector is non-zero)

$\Rightarrow \{v_1\}$ linearly indep.

$\boxed{\text{Induction argument}}$: Assume theorem holds for $k-1$ distinct eigenvalues.

Now, suppose $\vec{0} \neq v_i \in E_{\lambda_i}$, $i = 1, \dots, k$. We need to show that

$$\{v_1, \dots, v_k\} \text{ is linearly independent.} \qquad \text{——} \quad (*)$$

i.e. Assume $\exists\, a_i \in \mathbb{F}$ s.t.

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \vec{0} \qquad \text{——} \quad (1)$$

<u>Claim</u>: $\boxed{a_1 = a_2 = \cdots = a_k = 0}$ $\quad (\Rightarrow (*))$.

Apply T onto both sides of (1), using linearity of T,

$$a_1 T v_1 + a_2 T v_2 + \cdots + a_k T v_k = \vec{0}$$

$v_i \in E_{\lambda_i} \Rightarrow$

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \cdots + \boxed{a_k \lambda_k v_k} = \vec{0} \qquad \text{——} \quad (2)$$

$\boxed{T v_i = \lambda_i v_i}$ On the other hand, if we multiply (1) by λ_k:

$$a_1 \lambda_k v_1 + a_2 \lambda_k v_2 + \cdots + \boxed{a_k \lambda_k v_k} = \vec{0} \qquad \text{——} \quad (3)$$

Subtract the two equations (2) − (3) :

$$a_1(\lambda_1-\lambda_k)v_1 + a_2(\lambda_2-\lambda_k)v_2 + \cdots + a_{k-1}(\lambda_{k-1}-\lambda_k)v_{k-1} = \vec{0}$$

Since $\{v_1, \ldots, v_{k-1}\}$ is linearly indep. by induction hypothesis,

$$a_1(\lambda_1-\lambda_k) = a_2(\lambda_2-\lambda_k) = \cdots = a_{k-1}(\lambda_{k-1}-\lambda_k) = 0$$

As λ_i's are distinct, we have

$$a_1 = a_2 = \cdots = a_{k-1} = 0.$$

Putting this back to (1), $a_k v_k = \vec{0} \Rightarrow a_k = 0$. $(\because v_k \neq \vec{0})$

Therefore, $\{v_1, \ldots, v_k\}$ is linearly indep. and this proves the Theorem for any $k \in \mathbb{N}$ by induction.

$\square$

The Theorem above has the following very useful Corollary.

1st Sufficient Test for Diagonalizability:

If $A \in M_{n\times n}(\mathbb{F})$ has n distinct eigenvalues (in $\mathbb{F}$), then A is diagonalizable (over $\mathbb{F}$).

Example: Is $A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ diagonalizable?

Solution: YES! The characteristic polynomial is

$$f(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

Setting $f(\lambda)=0 \Rightarrow$ Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

A is 3×3 with 3 distinct eigenvalues $\Longrightarrow$ A diagonalizable!

$\square$

Note: To find an eigenbasis, we still have to go through the whole "flow chart".

From the example above, we can indeed generalize to make the following observation:

Prop: The eigenvalues of an upper (or lower) triangular matrix, i.e. $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ or $\begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$, are given by its diagonal entries.

Proof: Exercise!

Proof of 1st sufficient test:

$$\left.\begin{array}{llllll} \text{(distinct) eigenvalues:} & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \text{eigenvectors:} & v_1 & v_2 & \cdots & v_n \end{array}\right\} \xRightarrow{\text{Thm.}} \beta = \{v_1, \dots, v_n\} \text{ linearly indep.}$$

$$\Downarrow \dim \mathbb{F}^n = n$$

β is a **basis**, hence an **eigenbasis**. ∎

! TRAP: 1st sufficient test is **NOT** necessary!

i.e. A diagonalizable $\not\Rightarrow$ n distinct eigenvalues.

or equivalently, $\nexists$ n distinct eigenvalues $\not\Rightarrow$ A **not** diagonalizable.

Examples:

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ diagonalizable with only 1 eigenvalue $\lambda = 1$.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **not** diagonalizable with only 1 eigenvalue $\lambda = 1$.

Given $A \in M_{n\times n}(\mathbb{F})$ $\longrightarrow$ $\exists$ n distinct eigenvalues?

YES. $\rightarrow$ A diagonalizable

NO. $\rightarrow$ I dunno (yet)!

Thus, 1^{st} Sufficient Test is a quick and simple test, but it only works sometimes.

There is another useful quick test.

2^{nd} Sufficient Test for Diagonalizability:

If $A \in M_{n\times n}(\mathbb{R})$ is symmetric $(A^t = A)$, then A is diagonalizable (over $\mathbb{R}$)

Example: $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ is diagonalizable.

!TRAP: Again, not necessary:

e.g. $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is NOT symmetric but still diagonalizable! (why?)

Note:
- There is a corresponding version for $\mathbb{C}$ matrices.
- We will omit the proof for now. In fact this is a special case of the more general "spectral theorem", which is a main result in Ch. 6 of the textbook, after we have introduced the concept of "inner product space".

Q: Do we have a necessary and sufficient condition for diagonalizability?

Roughly speaking, A is diagonalizable if and only if there are "enough" eigenvectors (and eigenvalues).

We now go into more detail what does "enough" mean.

First, having "enough" eigenvalues means that the char. equation $\boxed{f(\lambda)=0}$ (polynomial equation of degree n) is "fully solvable", i.e. $\exists$ n roots (not nec. distinct!). In other words,

Lemma: If $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable (over $\mathbb{F}$), then the characteristic polynomial $f(\lambda)$ of A splits over $\mathbb{F}$, i.e.

$$f(\lambda) = c\,(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

fully "factorized" in $\mathbb{F}$

for some $c, \underbrace{\lambda_1,\dots,\lambda_k}_{\text{distinct}} \in \mathbb{F}$ s.t. $\boxed{m_1+m_2+\cdots+m_k = n}$

We call m_i the (algebraic) multiplicity of λ_i.

Example: $f(\lambda) = -(\lambda-1)^2(\lambda-2)$ splits (over $\mathbb{R}$ or $\mathbb{C}$)

$f(\lambda) = \lambda^2+1$ does not split over $\mathbb{R}$

$= (\lambda+i)(\lambda-i)$ but splits over $\mathbb{C}$

Remark: Any polynomial over $\mathbb{C}$ splits by the Fundamental Theorem of Algebra!

Proof of Lemma: Since A is diagonalizable, it is similar to a diagonal matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$. As similar matrices have the same char. polynomial $f(\lambda)$, it suffices to show that the char. poly. of D splits, which is true since

$$f(\lambda) = \det(D-\lambda I) = (d_1-\lambda)(d_2-\lambda)\cdots(d_n-\lambda)$$

(collecting like-terms) $\quad = (-1)^n(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$ splits!!

$\square$

Example: Given $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$, $(\mathbb{F} = \mathbb{R})$.

Is A diagonalizable? If so, find an invertible $Q \in M_{3\times3}(\mathbb{R})$ s.t. $Q^{-1}AQ$ is diagonal.

Solution: [A not symmetric. $\Rightarrow$ Cannot apply 2^{nd} Sufficient Test.]

char. poly. $= f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$

$$= (4-\lambda)\det\begin{pmatrix} 3-\lambda & 2 \\ 0 & 4-\lambda \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 2 & 3-\lambda \\ 1 & 0 \end{pmatrix}$$

$$= (4-\lambda)\left[(3-\lambda)(4-\lambda)\right] - (3-\lambda)$$

$$= (3-\lambda)\left[(4-\lambda)^2 - 1\right] = (3-\lambda)(\lambda^2 - 8\lambda + 15)$$

$$= -(\lambda-3)^2(\lambda-5) \qquad \text{splits!}$$

Set $f(\lambda) = 0 \Rightarrow$ $\boxed{\begin{array}{lll} \text{Eigenvalues:} & \lambda_1 = 3, & \lambda_2 = 5 \\ \text{multiplicity:} & m_1 = 2, & m_2 = 1 \end{array}}$

[No $\underline{\underline{3}}$ distinct eigenvalues $\Rightarrow$ 1^{st} Sufficient Test does NOT apply.]

Finding eigenspaces

$\lambda_1 = 3$, $E_{\lambda_1} = N(A - \lambda_1 I) = N\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$$\boxed{\dim E_{\lambda_1} = 2 = m_1}$$

$\lambda_2 = 5$, $E_{\lambda_2} = N(A - \lambda_2 I) = N\begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

$$\boxed{\dim E_{\lambda_2} = 1 = m_2}$$

Take $\underbrace{\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}}_{\text{eigenbasis}}$ or $Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow \underbrace{Q^{-1}AQ = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 5 \end{pmatrix}}_{\text{diagonalizable!}}$

$\square$

Thm: (Necessary & Sufficient Condition for Diagonalizability)

$A \in M_{n\times n}(\mathbb{F})$ is diagonalizable (over $\mathbb{F}$)

$\Longleftrightarrow$ (i) The char. polynomial $f(\lambda)$ of A splits (over $\mathbb{F}$).

(ii) For each eigenvalue $\lambda \in \mathbb{F}$ of A, $\dim E_\lambda = m_\lambda$

Example: Is $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ diagonalizable over $\mathbb{R}$?

Solution: upper triangular $\Rightarrow$ Eigenvalues : $\lambda_1 = 3$, $\lambda_2 = 4$

$f(\lambda) = \underbrace{(3-\lambda)^2(4-\lambda)}_{\text{splits!}}$ multiplicity : $m_1 = 2$, $m_2 = 1$

Compute eigenspaces: $E_{\lambda_1} = N(A - \lambda_1 I) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$$\dim E_{\lambda_1} = 1 \overset{!}{<} 2 = m_1$$

$\Rightarrow$ A NOT diagonalizable.

$\square$

Example: Is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ diagonalizable over $\mathbb{R}$? over $\mathbb{C}$?

Solution: Char. poly $= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = f(\lambda)$

NOT split over $\mathbb{R}$

$\Rightarrow$ A NOT diagonalizable over $\mathbb{R}$

However, $f(\lambda) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$ splits over $\mathbb{C}$ (always)

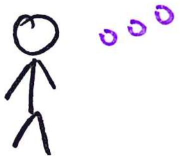
with 2 distinct eigenvalues $\lambda_1 = -i$, $\lambda_2 = i$

$\Rightarrow$ A diagonalizable over $\mathbb{C}$.

$\square$

An important application: $\boxed{\text{Finding } A^k}$

Example: Find A^{100} where $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$.



- If A were diagonal, say $A = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, then

$$A^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix} \quad \forall k \geq 1 \quad \text{(verify this.)}$$

- If A is just diagonalizable, i.e. $\exists Q$ s.t.

$$Q^{-1} A Q = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} = D$$

then $A = QDQ^{-1}$ and thus

$$A^2 = (QDQ^{-1})(QDQ^{-1}) = QD^2Q^{-1}$$

$$\vdots$$

$$\boxed{A^k = QD^kQ^{-1}} \quad \text{where } D^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$$

Solution: $\boxed{\text{Step 1}}$: Diagonalize A first (if possible)

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = -\lambda(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2 = 0$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$ all distinct $\Rightarrow$ diagonalizable!

Eigenspaces: $E_{\lambda_1} = N(A - I) = N\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

$E_{\lambda_2} = N(A - 2I) = N\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

Take $Q = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, then $Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = D$

$\boxed{\text{Step 2}}$: Raise powers of D then "conjugate" back:

$$A^{100} = QD^{100}Q^{-1} = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}}_{\begin{pmatrix} -2 & 2^{100} \\ 1 & -2^{100} \end{pmatrix}}\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2-2^{100} & 2-2^{100} \\ -1+2^{100} & -2+2^{101} \end{pmatrix}$$