Introduction to Game Theory

Lau Chi Hin

The Chinese University of Hong Kong

A simple war game

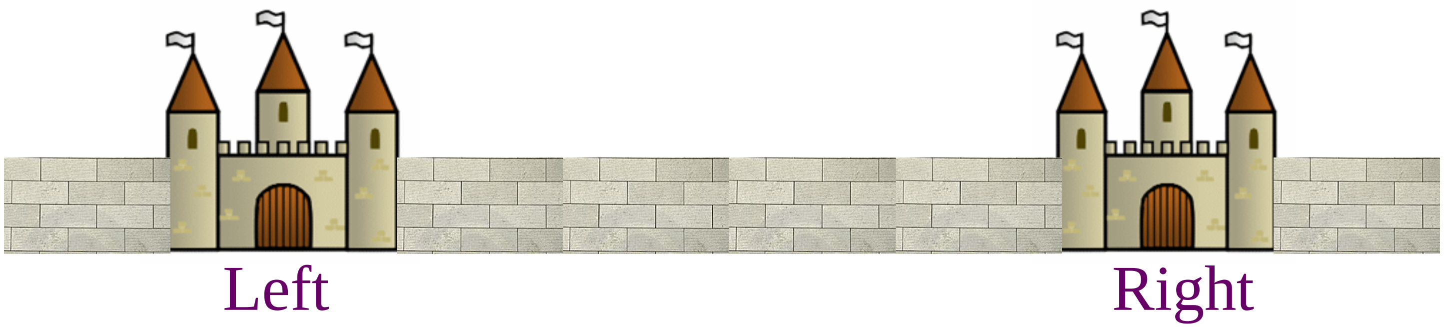
- **Two players: Attacker and Defender.**
- **A war at a castle which has two gates.**
- **Attacker: 2 troops ; Defender: 3 troops.**
- **Attacker and Defender may send each of their troops to either one of the gates.**

A simple war game

- **At each gate, the player with more troops wins at that gate. If the number of troops are equal, then Defender wins at that gate.**
- **Attacker wins if he wins at one of the gates.**
- **Defender wins if he wins both gates.**

A simple war game

Defender



Attacker

Who has an advantage to win?

Matrix representation of a game

Defender

Attacker		(3,0)	(2,1)	(1,2)	(0,3)
	(2,0)	D	D	A	A
	(1,1)	A	D	D	A
	(0,2)	A	A	D	D

Matrix representation of a game

Defender

		~~(3,0)~~	(2,1)	(1,2)	~~(0,3)~~
	(2,0)	~~D~~	D	A	~~A~~
Attacker	(1,1)	~~A~~	D	D	~~A~~
	(0,2)	~~A~~	A	D	~~D~~

Matrix representation of a game

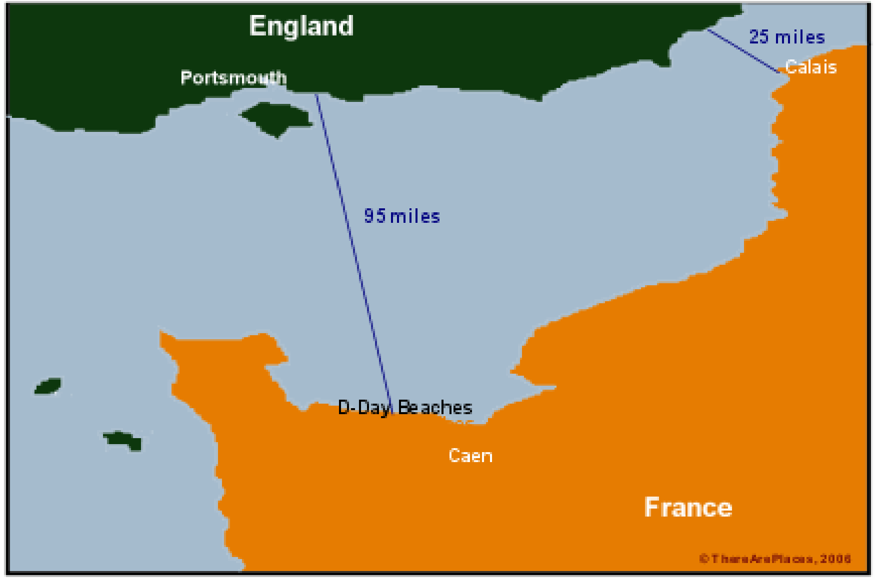
Defender

Attacker

	~~(3,0)~~	(2,1)	(1,2)	~~(0,3)~~
(2,0)	~~D~~	D	A	~~A~~
(1,1)	~~A~~	~~D~~	~~D~~	~~A~~
(0,2)	~~A~~	A	D	~~D~~

Invasion of Normandy

6 June 1944



The D-Day Beaches In Relationship to England

Normandy vs Calais

Colonel Blotto game

Colonel Blotto was tasked to distribute his n troops over k battle-fields knowing that on each battlefield the party that has allocated the most troops will win and the payoff is the number of winning field minus the number of losing field.

US presidential election 2000

George Bush
Republican

vs.

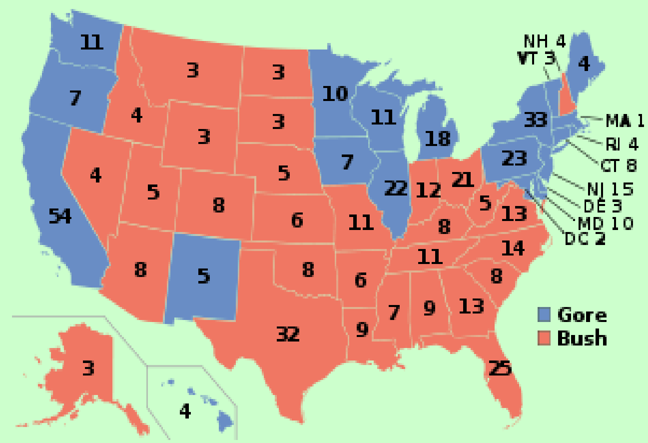
Al Gore
Democratic

US presidential election 2000

Merolla, Munger and Tofias,
Lotto, Blotto, or Frontrunner:
The 2000 U.S. Presidential Election and
The Nature of "Mistakes"

George Bush
50,456,002

Al Gore
50,999,897



271:266

Prisoner's dilemma

- John and Peter have been arrested for possession of guns. The police suspects that they are going to commit a major crime.
- If no one confesses, they will both be jailed for 1 year.
- If only one confesses, he'll go free and his partner will be jailed for 5 years.
- If they both confess, they both get 3 years.

Prisoner’s dilemma

		Peter	
		Confess	Don’t confess
John	Confess	(-3,-3)	(0,-5)
	Don’t confess	(-5,0)	(-1,-1)

Prisoner's dilemma

		Peter	
		Confess	Don't
John	Confess	(-3,-3)	(0,-5)
	Don't	(-5,0)	(-1,-1)

- If Peter confesses：

  John "confess" (3 years) better than "don't confess" (5 years).
- If Peter doesn't confess：

  John "confess" (0 year) better than "don't confess" (1 year).

Prisoner's dilemma

		Peter	
		Confess	Don't
John	Confess	(-3,-3)	(0,-5)
	Don't	(-5,0)	(-1,-1)

- Thus John should confess whatever Peter does.
- Similarly, Peter should also confess.

Conclusion: Both of them should confess

Prisoner's dilemma

		Peter	
		Confess	Don't confess
John	Confess	(-3,-3)	(0,-5)
	Don't confess	(-5,0)	(-1,-1)

Applications

- Economics
- Political science
- Ecology
- Computer science

Vickrey auction

The highest bidder wins, but the price paid is the second-highest bid.

Vickrey auction

明 報 **2009年10月28日**
再論以博弈論打破勾地困局

政府可考慮，如勾地者最終成功投得地皮，可讓他們享有3至5%的折扣優惠，如此建議獲接納，發展商會甘心做「出頭鳥」，搶先以高價勾地。

…其他發展商，如出價不及勾出地皮的發展商，已考慮了市場情況和財政計算，他們亦知其中一個對手享有折扣優惠，所以要打敗對手，出價只有更進取。…

也可考慮將最終成交價訂為拍賣地皮的第二最高出價。」

撰文:陸振球 (明報地產版主管)

All pay auction

In an all pay auction, every bidder pays what they bid regardless of whether or not they win.

Examples:

- Elections
- Sports competitions
- Wars

Shubik's dollar auction

The auctioneer auctions off a dollar bill to the highest bidder, with the understanding that both the highest bidder and the second highest bidder will pay.

Martin Shubik: The dollar auction game: A paradox in noncooperative behavior and escalation, Journal of Conflict Resolution, Vol. 15 (1971)

Doll crane machine

Attempting to reduce the loss by continuing to play

Nobel laureates related to game theory

- 1994: Nash, Harsanyi, Selten
- 1996: Vickrey
- 2005: Aumann, Schelling
- 2007: Hurwicz, Maskin, Myerson
- 2012: Shapley, Roth
- 2014: Tirole

Price war

Two supermarkets PN and WC are engaging in a price war.

vs

Price war

- Each supermarket can choose: high price or low price.
- If both choose high price, then each will earn $4 (million).

Price war

- If both choose low price, then each will earn $2 (million).
- If they choose different strategies, then the supermarket choosing high price will earn $0 (million), while the one choosing low price will earn $5 (million).

Price war

		WC	
		Low	High
PN	Low	(2,2)	(5,0)
	High	(0,5)	(4,4)

Price war

		WC	
		Low	High
PN	Low	(2,2)	~~(5,0)~~
	High	~~(0,5)~~	~~(4,4)~~

Price war vs Prisoner dilemma

		Peter	
		Confess	Don’t
John	Confess	(-3,-3)	(0,-5)
	Don’t	(-5,0)	(-1,-1)

		WC	
		Low	High
PN	Low	(2,2)	(5,0)
	High	(0,5)	(4,4)

These are called **dominant strategy equilibrium.**

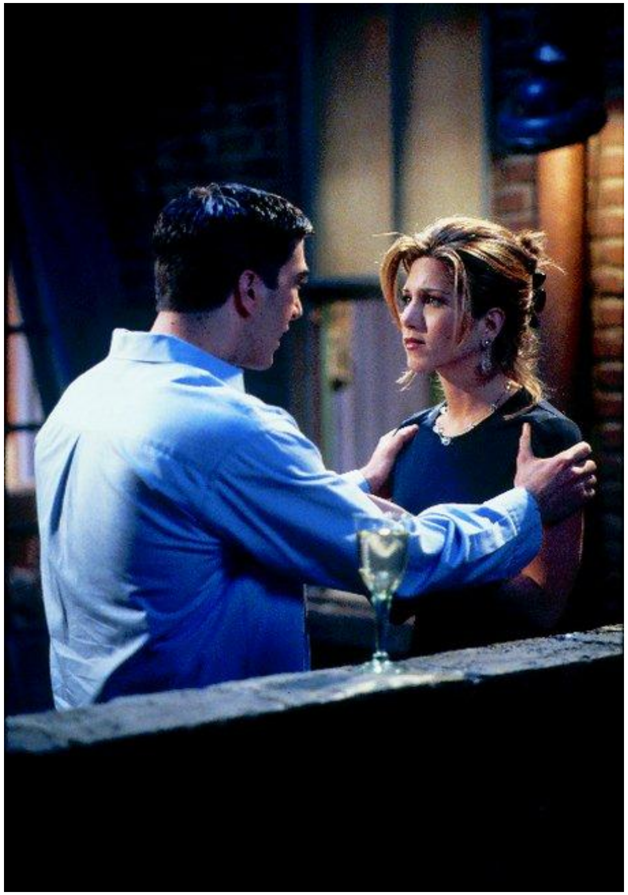
Dominant strategy equilibrium

- A strategy of a player is a dominant strategy if the player has the best return no matter how the other players play.
- If every player chooses its dominant strategy, it is called a dominant strategy equilibrium.

Dominant strategy equilibrium

- Not every game has dominant strategy equilibrium.
- A player of a game may have no dominant strategy.

Dating game



Roy and Connie would like to go out on Friday night.

Roy prefers to see football, while Connie prefers to have a drink.

However, they would rather go out together than be alone.

Dating game

		Connie	
		Football	Drink
Roy	Football	(20,5)	(0,0)
	Drink	(0,0)	(5,20)

Both Roy and Connie do not have dominant strategy. Therefore dating game does not have dominant strategy equilibrium.

Pure Nash equilibrium

- A choice of strategies of the players is a **pure Nash equilibrium** if no player can increase its gain given that *all other players do not change their strategies*.
- A dominant strategy equilibrium is always a pure Nash equilibrium.

Pure Nash equilibrium

Prisoner's dilemma

		Peter	
		Confess	Don't
John	Confess	(-3,-3)	(0,-5)
	Don't	(-5,0)	(-1,-1)

Prisoner dilemma has a pure Nash equilibrium because it has a dominant strategy equilibrium.

Pure Nash equilibrium

Dating game

		Connie	
		Football	Drink
Roy	Football	(20,5)	(0,0)
	Drink	(0,0)	(5,20)

Dating game has no dominant strategy equilibrium but has two pure Nash equilibria.

Rock-paper-scissors

		Column player		
		Rock	Paper	Scissor
Row player	Rock	(0,0)	(-1,1)	(1,-1)
	Paper	(1,-1)	(0,0)	(-1,1)
	Scissor	(-1,1)	(1,-1)	(0,0)

Rock-paper-scissors has no pure Nash equilibrium.

Mixed strategy

Pure strategy

Using one strategy constantly.

Mixed strategy

Using varies strategies according to certain probabilities.

(Note that a pure strategy is also a mixed strategy where one of the strategies is used with probability 1 and all other strategies are used with probability 0.)

Mixed Nash equilibrium

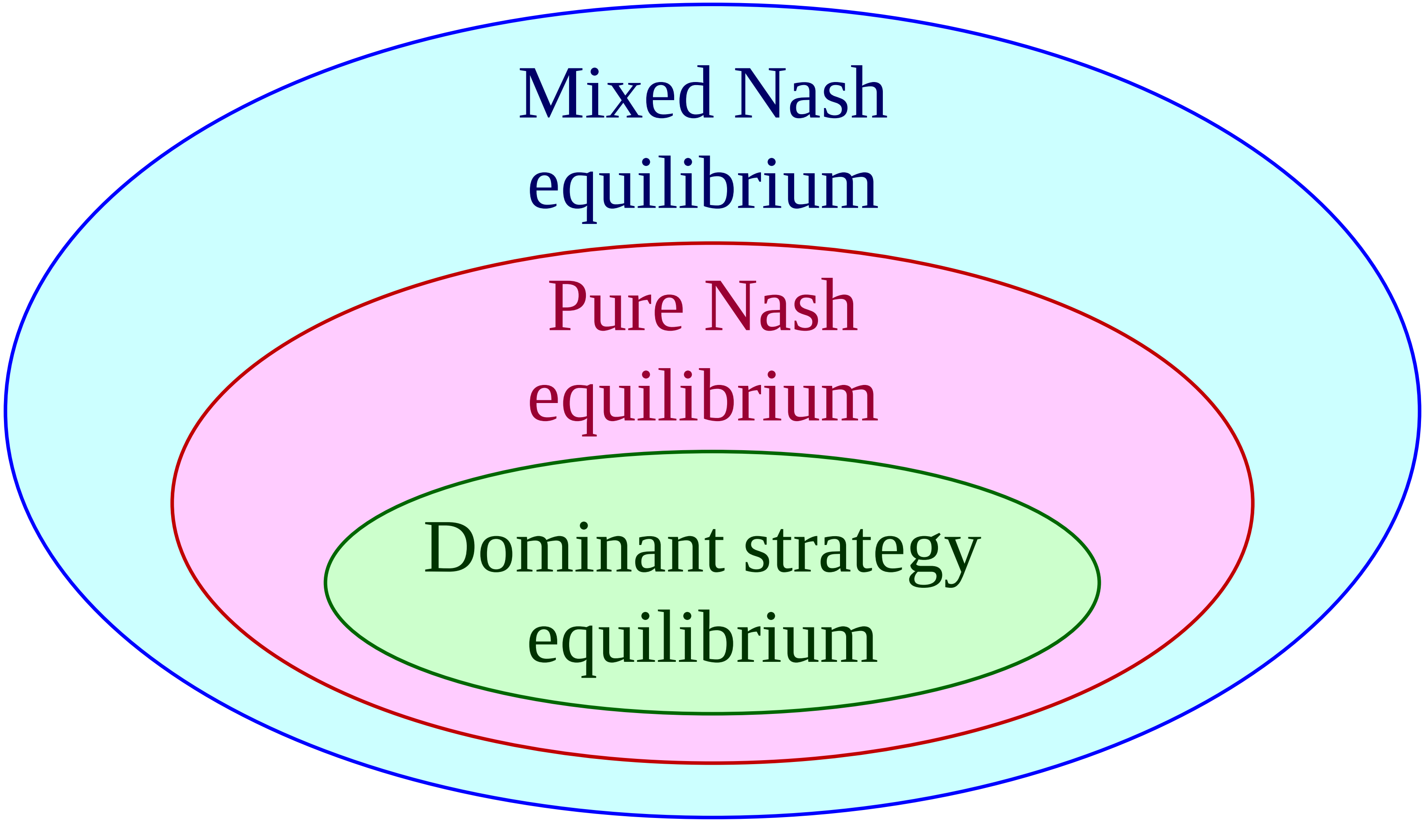
- A choice of mixed strategies of the players is called a mixed Nash equilibrium if no player has anything to gain by changing his own strategy alone while all other players do not change their strategies.
- We will simply call a mixed Nash equilibrium Nash equilibrium.

Rock-paper-scissors

		Column player		
		Rock	Paper	Scissor
Row player	Rock	(0,0)	(-1,1)	(1,-1)
	Paper	(1,-1)	(0,0)	(-1,1)
	Scissor	(-1,1)	(1,-1)	(0,0)

A mixed Nash equilibrium is both players use mixed strategy (1/3,1/3,1/3), that means all three gestures are used with the same probability 1/3.

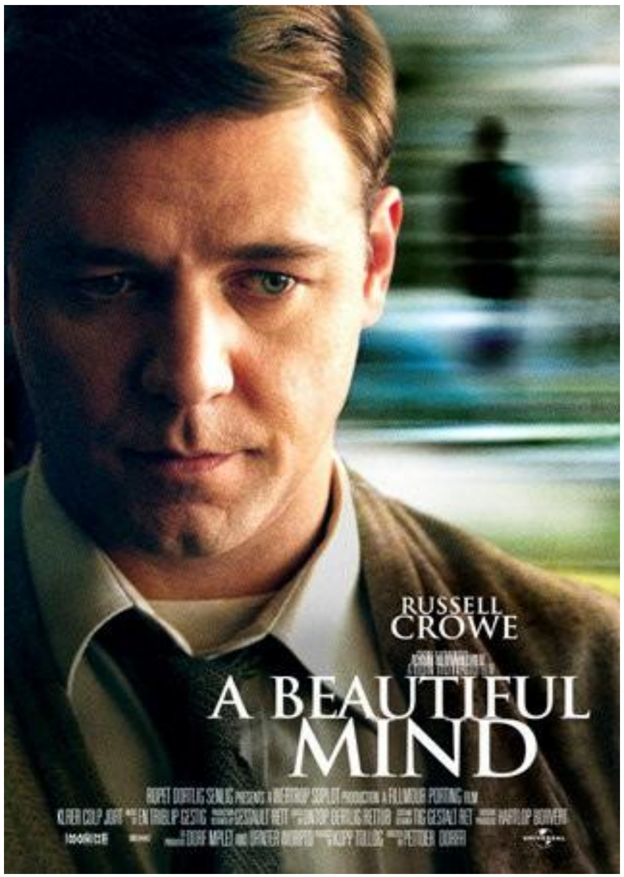
Mixed Nash equilibrium

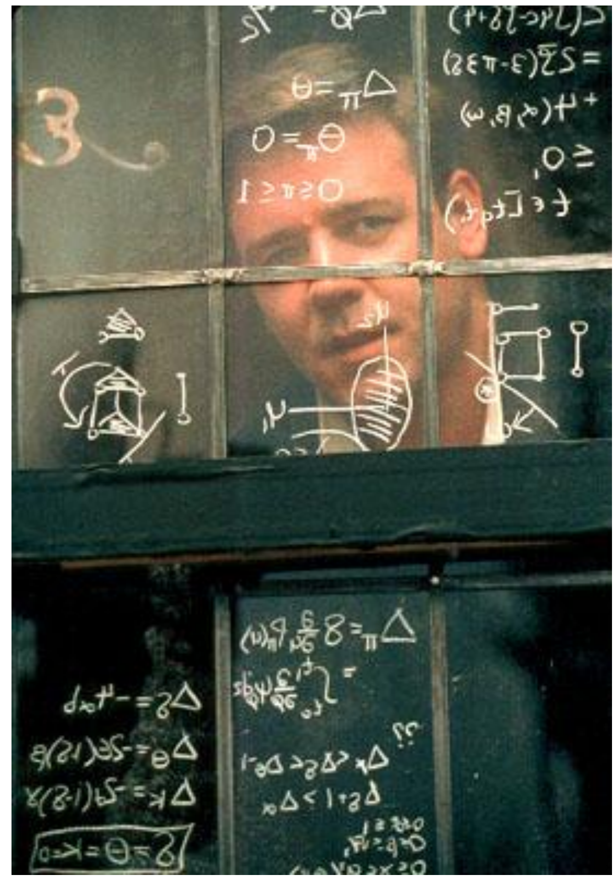


Mixed Nash equilibrium

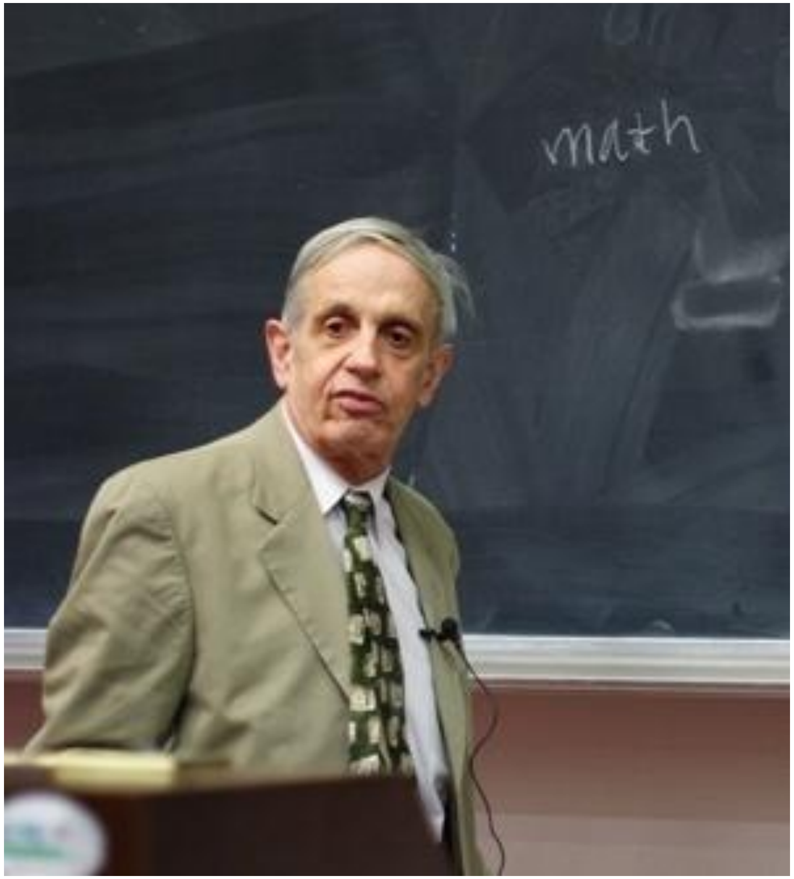
Game	Dominant strategy equilibrium	Pure Nash equilibrium	Mixed Nash equilibrium
Prisoner's dilemma	✓	✓	✓
Dating game	✗	✓	✓
Rock-paper-scissors	✗	✗	✓

A Beautiful Mind





John Nash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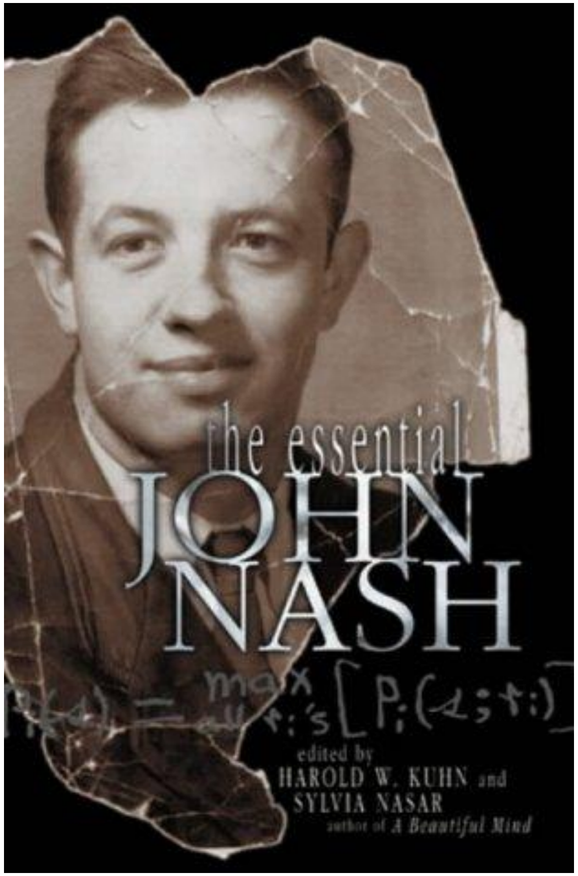
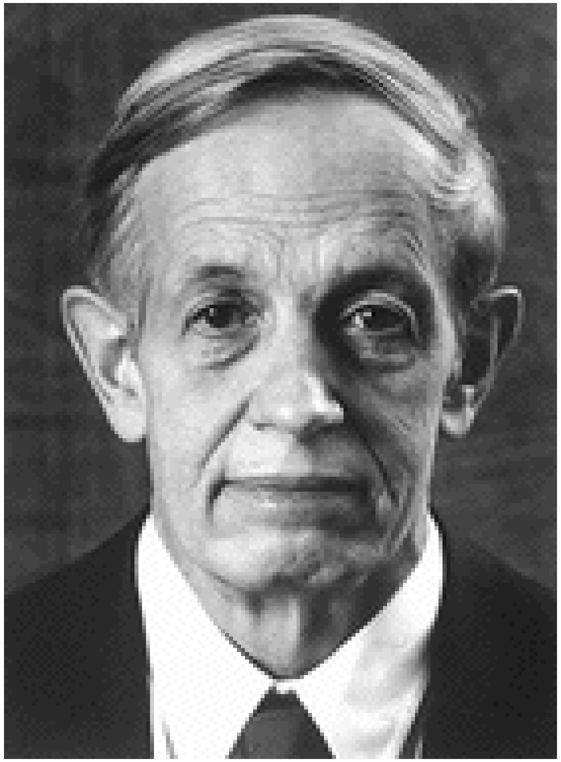

the essential
JOHN
NASH
edited by
HAROLD W. KUHN and
SYLVIA NASAR
author of A Beautiful Mind

John Nash

- Born in 1928
- Earned a PhD from Princeton at 22



John Nash

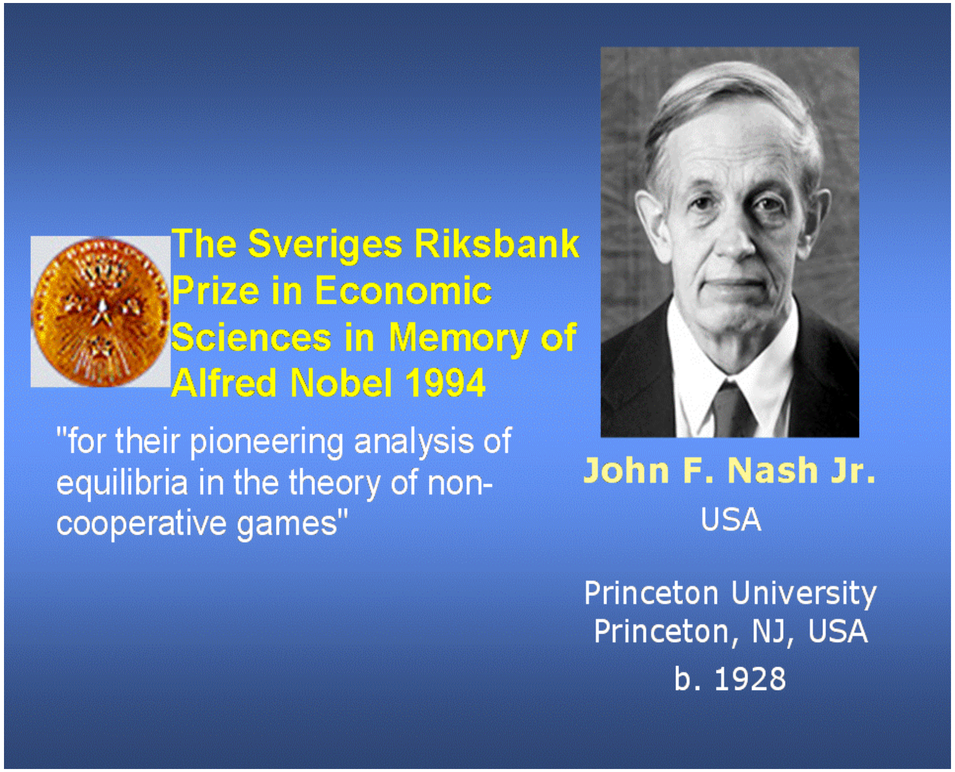
- Married Alicia Larde, a physics student at MIT, in 1957

John Nash

- Late 1950s, Nash left MIT because of mental illness.
- It is a miracle that he can recover twenty years later.

John Nash

- In 1994, Nash shared the Nobel Prize in Economics with John C. Harsanyi and Reinhard Selten



Nash equilibrium

Nash equilibrium in the movie

http://www.youtube.com/watch?v=uAJDD1_Oexo

Nash equilibrium

The example in the movie is not a Nash equilibrium.

Nash’s Theorem

John Nash (Annals of math 1957)

Theorem: Every finite *n*-player non-cooperative game has a mixed Nash equilibrium.

Modified rock-paper-scissors

		Column player	
		Rock	Scissor
Row player	Rock	(0,0)	(1,-1)
	Paper	(1,-1)	(-1,1)

What is the mixed Nash equilibrium?

Modified rock-paper-scissors

		Column player	
		Rock	Scissor
Row player	Rock	(0,0)	(1,-1)
	Paper	(1,-1)	(-1,1)

Mixed Nash equilibrium:

Row player: (2/3,1/3)

Column player: (2/3,1/3)

A 3-person game

- Each one can show either one finger or two fingers.
- If you are the only one showing one finger, then you get one dollar.
- If you are the only one showing two fingers, then you get two dollars.

- Otherwise, everyone gets nothing !
- Can you find a Nash equilibrium for this game ?

A 3-person game

A mixed Nash equilibrium is roughly 41% of using one finger and 59% of using two fingers for each player.

Nash's Proof

Brouwer fixed-point theorem

pieces of algebraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

A proof of this existence theorem based on Kakutani's generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U. S. A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem. We proceed by constructing a continuous transformation T of the space of n-tuples such that the fixed points of T are the equilibrium points of the game.

THEOREM 1. *Every finite game has an equilibrium point.*

PROOF. Let $\mathfrak{s}$ be an n-tuple of mixed strategies, $p_i(\mathfrak{s})$ the corresponding pay-off to player i, and $p_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he changes to his α^{th} pure strategy $\pi_{i\alpha}$ and the others continue to use their respective mixed strategies from $\mathfrak{s}$. We now define a set of continuous functions of $\mathfrak{s}$ by

$$\varphi_{i\alpha}(\mathfrak{s}) = \max(0, p_{i\alpha}(\mathfrak{s}) - p_i(\mathfrak{s}))$$

and for each component s_i of $\mathfrak{s}$ we define a modification s_i' by

$$s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})},$$

calling $\mathfrak{s}'$ the n-tuple $(s_1', s_2', s_3' \cdots s_n')$.

We must now show that the fixed points of the mapping $T: \mathfrak{s} \to \mathfrak{s}'$ are the equilibrium points.

First consider any n-tuple $\mathfrak{s}$. In $\mathfrak{s}$ the i^{th} player's mixed strategy s_i will use certain of his pure strategies. Some one of these strategies, say $\pi_{i\alpha}$, must be "least profitable" so that $p_{i\alpha}(\mathfrak{s}) \leqq p_i(\mathfrak{s})$. This will make $\varphi_{i\alpha}(\mathfrak{s}) = 0$.

Now if this n-tuple $\mathfrak{s}$ happens to be fixed under T the proportion of $\pi_{i\alpha}$ used in s_i must not be decreased by T. Hence, for all β's, $\varphi_{i\beta}(\mathfrak{s})$ must be zero to prevent the denominator of the expression defining s_i' from exceeding 1.

Thus, if $\mathfrak{s}$ is fixed under T, for any i and β $\varphi_{i\beta}(\mathfrak{s}) = 0$. This means no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. But this is just a criterion for an eq. pt. [see (2)].

Conversely, if $\mathfrak{s}$ is an eq. pt. it is immediate that all φ's vanish, making $\mathfrak{s}$ a fixed point under T.

Since the space of n-tuples is a cell the Brouwer fixed point theorem requires that T must have at least one fixed point $\mathfrak{s}$, which must be an equilibrium point.

Symmetries of Games

An *automorphism*, or *symmetry*, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

Brouwer's fixed-point theorem

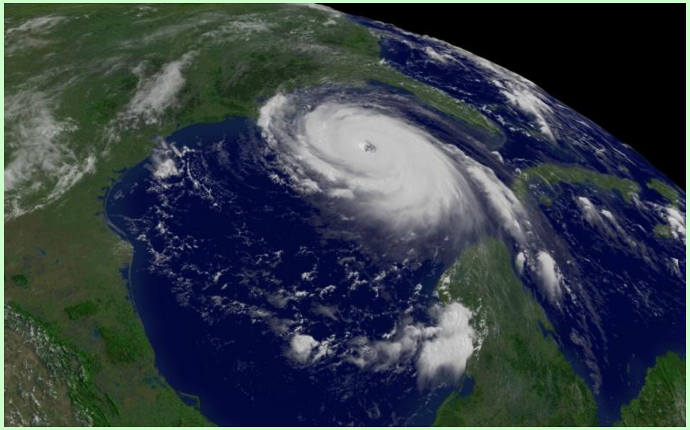
Fixed-point theorem:

Any continuous function from the *n*-dimensional closed unit ball to itself has at least one fixed-point.

Consequence of fixed-point theorem

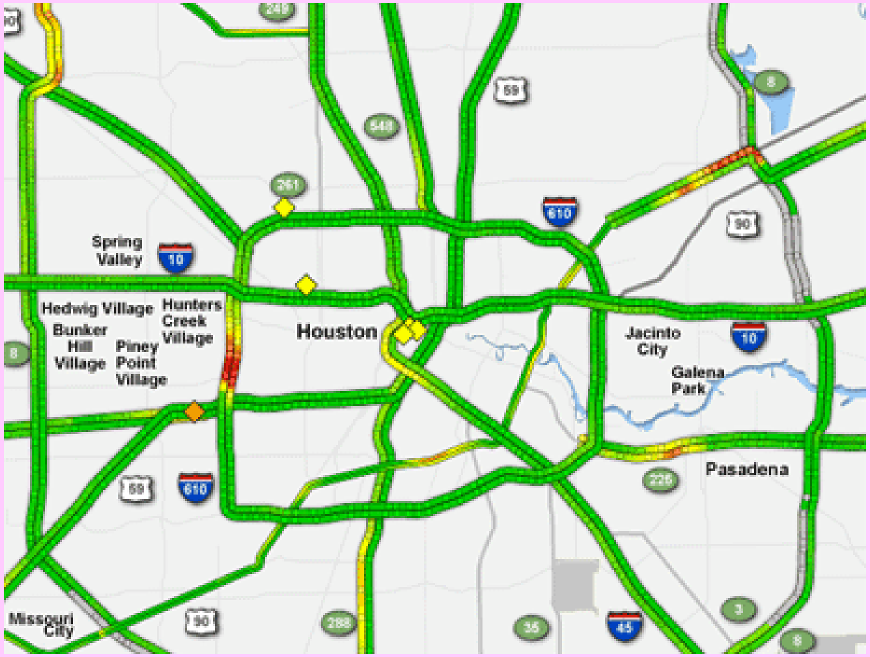
- Everybody has at least one bald spot.

- There is at least one place on earth with no wind.

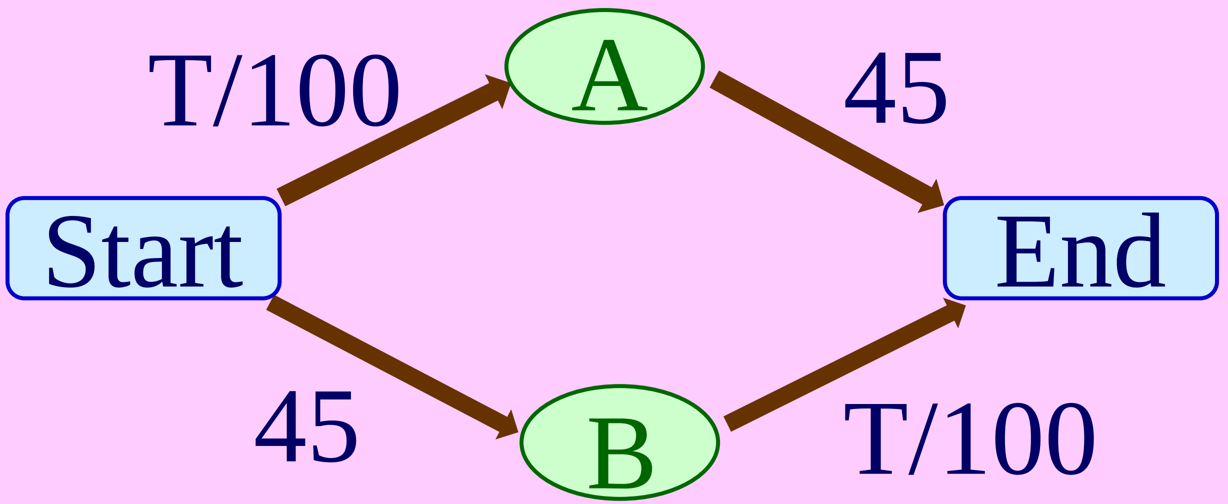


Braess paradox

Building a new road always good?



Braess paradox

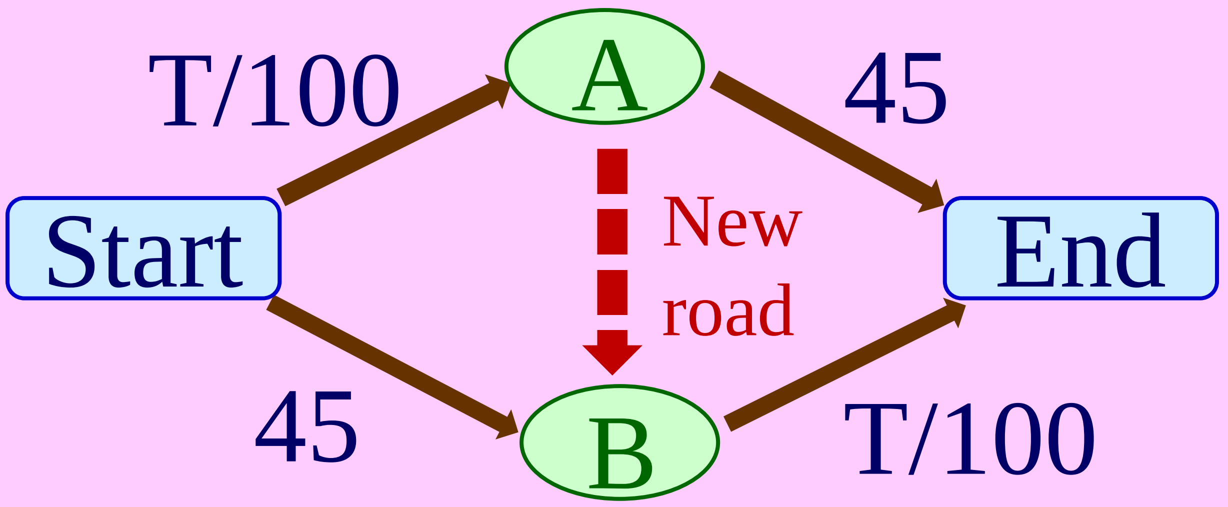


Number of vehicles:4000

Vehicles via A: 2000; Vehicles via B:2000

Expected time: 65 mins

Braess paradox

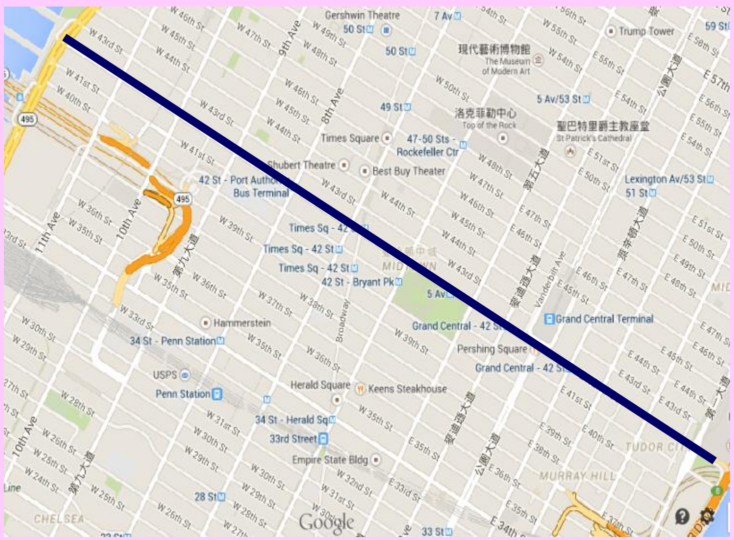


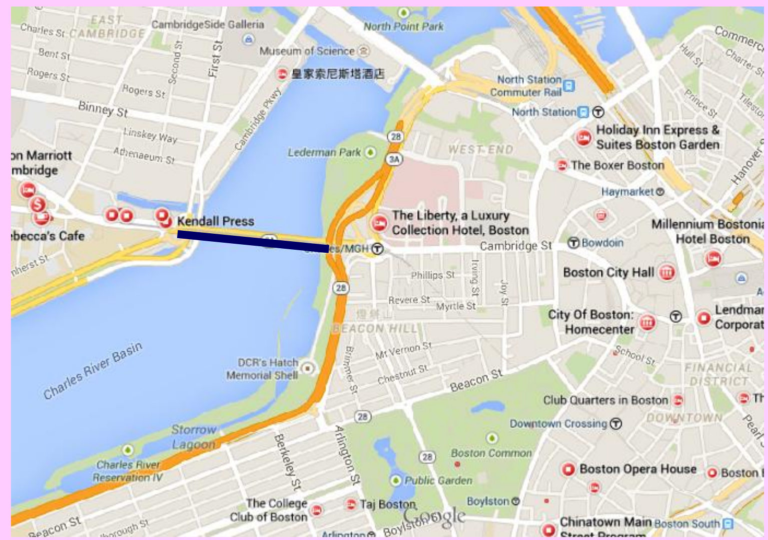
Number of vehicles:4000

All vehicles via A and B

Expected time: 80 mins

Braess paradox in traffic network





New York City

42nd Street

Boston

Main Street

Braess paradox in electric circuit

Transport inefficiency in branched-out mesoscopic networks: An analog of the Braess paradox: M.G.Pala and others, Physical Review Letters 108 (7), 2012

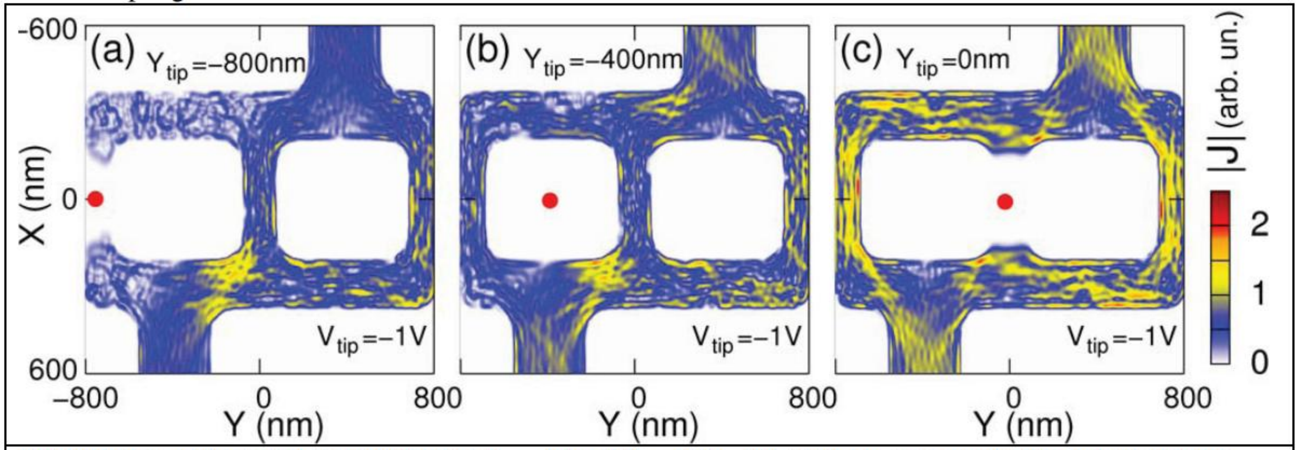


FIGURE 4 (a-c) Contour plots of |J|(*X,Y*) for V_{tip}= -1V and Y_{tip}= -800, -400 and 0 nm, respectively. The red circular dots point toward the tip position. The corral geometry, drain-source voltage and λ_F are identical to Fig. 3(a). The same color scale is used for all three images.

Traveler’s dilemma

Two travelers lost their suitcases. The airline manager asks them to write down the amount of the dollar value of the suitcases at no less than $2 and no larger than $100. If both write down the same number, the manager will treat that number as the true dollar value of both suitcases and reimburse both travelers that amount. However, if one writes down a smaller number than the other, this smaller number will be taken as the true dollar value, and both travelers will receive that amount along with a bonus: $2 extra will be paid to the traveler who wrote down the lower value and a $2 deduction will be taken from the person who wrote down the higher amount.

Traveler’s dilemma

Kauchik Basu,
"The Traveler's Dilemma: Paradoxes of Rationality in Game Theory";
American Economic Review, Vol. 84, No. 2, pages 391-395; May 1994.

Traveler's dilemma

		Billy				
		100	99	98	…	2
Alan	100	(100,100)	(97,101)	(96,100)	…	(0,4)
	99	(101,97)	(99,99)	(96,100)	…	(0,4)
	98	(100,96)	(100,96)	(98,98)	…	(0,4)
	…	…	…	…	…	…
	2	(4,0)	(4,0)	(4,0)	...	(2,2)

Traveler's dilemma

		Billy				
		100	99	98	…	2
Alan	100	(100,100)	(97,101)	(96,100)	…	(0,4)
	99	(101,97)	(99,99)	(96,100)	…	(0,4)
	98	(100,96)	(100,96)	(98,98)	…	(0,4)
	…	…	…	…	…	…
	2	(4,0)	(4,0)	(4,0)	…	(2,2)

Traveler’s dilemma

When the upper limit is 3, the Traveler’s dilemma is similar to Prisoner's dilemma

		Billy	
		3	2
Alan	3	(3,3)	(0,4)
	2	(4,0)	(2,2)

Traveler’s dilemma

		Peter	
		Don’t	Confess
John	Don’t	(-1,-1)	(-5,0)
	Confess	(0,-5)	(-3,-3)

Prisoner's dilemma