THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH3070 (Second Term, 2014–2015)

Introduction to Topology

Exercise 0 Preparation

Remarks

These exercises may give you an impression of the foundation needed in this course.

1. Write down precise and concise statements of the following.
   (a) The definition of the *continuity* of a function $f\colon\ X \subset \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in X$.
   (b) The definition of the limit of a function $f\colon\ X \subset \mathbb{R} \to \mathbb{R}$ as $x \to x_0 \in X$.
   (c) The relation between the continuity of a function $f\colon\ X \subset \mathbb{R} \to \mathbb{R}$ at $x_0 \in X$ and a sequence $x_n \to x_0 \in X$.
   (d) The definition of supremum and infinum of a set $A \subset \mathbb{R}$.
   (e) The definition of supremum and infinum of a subset A in a partially ordered set X.
2. Let $f\colon\ X \to Y$ and $g\colon\ Y \to Z$; $A \subset X$, $B \subset Y$; if needed, $f(A) \subset B$. Determine the correctness of the following statements. Justify with proofs or counter-examples.
   (a) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$
   (b) if $B_1 \subset B_2$ then $f^{-1}(B_1) \subset f^{-1}(B_2)$
   (c) if $A_1 \subset A_2$ then $f(A_2 * A_1) = f(A_2) * f(A_1)$ where $*$ may be $\cup$, $\cap$, $\setminus$ (set minus), or $\triangle$ (symmetric difference).
3. Define a relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 - y_1^2 = x_2^2 - y_2^2$. Show that this is an equivalence relation. What are its equivalence classes?
4. Define a family of sets X_α for $\alpha \in A$ (A may be infinite). Then define the arbitrary product $\prod_{\alpha \in A} X_\alpha$.

   If there are functions $f_\alpha\colon\ X_\alpha \to Y$, is it possible to define a function $f\colon\ \prod_{\alpha \in A} X_\alpha \to Y$?

   On the other hands, if there are functions $g_\alpha\colon\ U \to X_\alpha$, is it possible to define a function $g\colon\ U \to \prod_{\alpha \in A} X_\alpha$?
5. Let $A_\alpha \subset X$ where $\alpha \in A$. Define $\bigcup_{\alpha \in A} A_\alpha$ and $\bigcap_{\alpha \in A} A_\alpha$.

   For $B \subset A$, what is the meaning of $\bigcup \{ A_\alpha : \alpha \in B \}$? What is the meaning of all arbitrary unions of sets in $\{ A_\alpha : \alpha \in A \}$?
6. What is a countable or uncountable set? State some propositions about countability between a set and its image under a function.
7. What are the basic requirements of an algebraic group?