connected: $\forall S \subset X$ which is both open and closed. $S=\phi$ or $S=X$.

pathwise connected: $\forall x_1, x_2 \in X$, $\exists$ continuous $\gamma: [0,1] \to X$ s.t. $\gamma(0)=x_1$, $\gamma(1)=x_2$

locally pathwise connected: $\forall x \in X$, $\exists$ nbd base $\beta_x = \{B \in \beta_x : B$ is path connected$\}$

1 Every path connected space is connected.

Suppose X is path connected. If $\{A, B\}$ disconnects X, then chose $a \in A$, $b \in B$, and a path $\gamma: [0,1] \to X$ from a to b.

$\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ disconnects $[0,1]$, a contradiction. Thus X is connected.

2. In a topological space it is impossible to join an interior point of a set to an exterior point without intersecting the boundary of the set

Suppose $\gamma: [0,1] \to X$ is a path and E is a set. such that

$\gamma(0) \in \text{Int} E$, $\gamma(1) \in X \setminus \bar{E}$. We want to show $\exists\, t_0 \in [0,1]$ s.t. $\gamma(t_0) \in \text{Bdry} E$

If $\forall t \in [0,1]$, $\gamma(t) \notin \text{Bdry} E$. i.e. $\gamma^{-1}(\text{Bdry} E) = \phi$

Since $X = (\text{Int} E) \cup (\text{Bdry} E) \cup (X \setminus \bar{E})$,

$[0,1] = \gamma^{-1}(\text{Int} E) \cup \gamma^{-1}(X \setminus \bar{E})$, a controdiction!

3. If a space X is connected and locally path connected, then X is path connected.

Fix $a \in X$. let $H = \{b \in X : \exists$ path in X from a to $b\}$

Since $a \in H$, we know that $H \neq \phi$. We will show that H is both open and closed, so that $H = X$. (X is connected).

Suppose $b \in H$. Let $U \in \beta_b$ and U is path connected.

For $\forall x \in U$, $\exists$ path connecting b to x and path from a to b. since $b \in H$.

Adding these two paths, we get a path from a to x.

Hence $U \subset H$. Thus H is open.

Suppose $x \in \bar{H}$. Let $U \in \mathcal{B}_x$ and U is path connected.

Since $U \cap H \neq \phi$, chose a point $b \in U \cap H$.

$\exists$ path from a to b. and path from b to x. $\Rightarrow \exists$ path from a to x.

Hence $\bar{H} = H$. Thus H is closed.

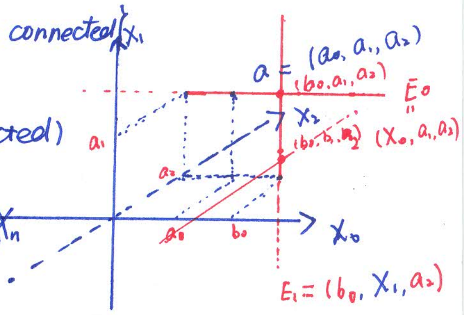
4. Product space $\prod_{n \in \omega} X_n \neq \phi$ is connected $\Leftrightarrow$ $\forall n \in \omega$. X_n is connected.

$\Rightarrow$ Each projection mapping is continuous and onto. $\pi_n(\prod_{n \in \omega} X_n) = X_n$ is connected

$\Leftarrow$ Choose a point $a \in \prod_{n \in \omega} X_n$.

Let ~~$E = \{x \in \prod_{n\in\omega} X_n : \}$~~ $E = \bigcup\{C \subset \prod_{n\in\omega} X_n : a \in C, C \text{ is connected}\}$

E be the component of a. i.e. ~~$E = \bigcup\{C \subset X; x \in$~~



We want to show $\bar{E} = \prod_{n \in \omega} X_n$. ($E$ connected $\Rightarrow \bar{E}$ connected)

Suppose that $U = \bigcap_{i=0}^{k} \pi_i^{-1}(U_i) \neq \phi$ is a basic open set in $\prod_{n\in\omega} X_n$

For $i = 0, \cdots, k$. choose $b_i \in U_i$. Define

$E_0 = \{c \in \prod_{n\in\omega} X_n : c_0$ is arbitrary and $c_n = a_n$ otherwise$\}$.

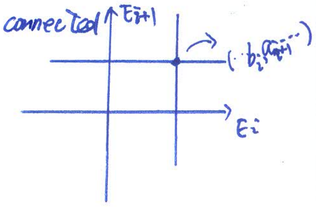
$E_i = \{c \in \prod_{n\in\omega} X_n : c_j = b_j$ for $j = 0, 1, \cdots, i-1$. c_i is arbitrary and $c_n = a_n$ otherwise$\}$.

$E_{i+1} = \{c \in \prod_{n\in\omega} X_n : c_j = b_j$ for $j = 0, 1, \cdots, i$, c_{i+1} is arbitrary and $c_n = a_n$ otherwise$\}$.

$E_k = \{c \in \prod_{n\in\omega} X_n : c_j = b_j$ for $j = 0, 1, \cdots k-1$, c_k is arbitrary and $c_n = a_n$ otherwise$\}$.

For each i, $E_i \cong X_i$ and hence E_i is connected for $i = 0, 1, \cdots, k$.

For each i $(b_0, \cdots, b_i, a_{i+1}, a_{i+2}, \cdots) \in E_i \cap E_{i+1} \neq \phi$, and thus $F = \bigcup_{i=0}^{k} E_i$ is connected



Since $a \in E_0 \subset F$, then $F \subset E$

Since $(b_0, \cdots, b_k, a_{k+1}, \cdots) \in E_k \cap U \neq \phi$, $\Rightarrow E \cap U \supset$ ~~F_k~~ $F \cap U \neq \phi$

Hence E intersects every nonempty basic open set, and thus $\bar{E} = \prod_{n\in\omega} X_n$

E connected, f ctns $\Rightarrow$ $f(E)$ connected

$E \subset A \subset \bar{E}$, E connected $\Rightarrow$ A connected. ($\bar{E}$ connected)