Tutorial

1. Let $(X/\sim, \mathcal{T}_q)$ be the quotient space of $(X, \mathcal{T})$ with an equivalence relation $\sim$. Denote the quotient map by $q: X \to X/\sim$.

   (a) If the space X is separable, is its quotient also separable? Justify your answer.

   (b) If the space X is Hausdorff, is its quotient also Hausdorff? Justify your answer.

(a). Yes. Let D be a countable dense set of X.

Consider $q(D)$ which is a countable set in $X/\sim$.

Now for $\forall$ non-empty open set U in $X/\sim$,

q is cts & surj $\Rightarrow$ $q^{-1}(U)$ is non-empty open in X

$\Rightarrow q^{-1}(U) \cap D \neq \emptyset \Rightarrow U \cap q(D) \neq \emptyset$

So $q(D)$ is also dense.

(b). No.

Let $X = \{(x, y) \mid x \in \mathbb{R}^1,\ y = 0 \text{ or } 1\} \subseteq \mathbb{R}^2$

with the induced topology as a subset of $(\mathbb{R}^2, \mathcal{T}_{std})$

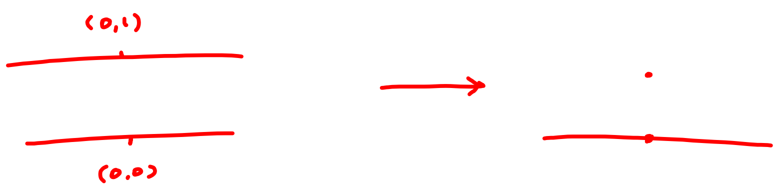
Define $\sim$ on X:

Glue $(x, 0)$ and $(x, 1)$ for $\forall\ x \neq 0$.

In $X/\sim$, we hope that:

For $\forall$ open nbd N_0 of $[(0,0)]$ $\forall$ open nbd N_1 of $[(0,1)]$

We have: $N_0 \cap N_1 \neq \emptyset$.



$q^{-1}(N_0)$ is an open nbd of $(0,0)$

$\Rightarrow \exists.\ \varepsilon_0 > 0$ st. $(-\varepsilon_0, \varepsilon_0) \times \{0\} \subseteq q^{-1}(N_0)$

Therefor $q\left((-\varepsilon_0, \varepsilon_0) \times \{0\}\right) \subseteq N_0$

Similarly, $\exists\, \varepsilon_1 > 0$, st. $q\left((-\varepsilon_1, \varepsilon_1) \times \{1\}\right) \subseteq N_1$

$$\Rightarrow \underbrace{q\left((-\varepsilon_0, \varepsilon_0) \times \{0\}\right) \cap q\left((-\varepsilon_1, \varepsilon_1) \times \{1\}\right)}_{\neq \emptyset} \subseteq N_1 \cap N_2$$

2. If X, Y are two topo spaces. $f: X \to Y$ is a quotient map. Let $R \triangleq \{(x,y) \in X \times X \mid f(x) = f(y)\}$

Show: (a) R is an equivalent relation in X

(b) Y is homeomorphism to X/R

Pf: (a) trivial.

(b). Define a map: $\tilde{f}: X/R \to Y$

$[x] \to f(x)$

- $\tilde{f}$ is well defined.
- inj.
- suj. f is quotient map.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] & \\ X/R \arrow[r, "\tilde{f}"'] & Y \end{tikzcd}$$

is a $\underline{\text{commutative}}$ digram

$f = \tilde{f} \circ q$

- $\tilde{f}$ is cts.

For $\forall$ open $O \subset Y$. We consider $\tilde{f}^{-1}(O)$.

$$\tilde{f}^{-1}(O) \text{ open} \overset{\text{quo topo}}{\Longleftrightarrow} \underline{q^{-1}(\tilde{f}^{-1}(O))} \text{ open}$$

$$(\tilde{f} \circ q)^{-1}(O) = f^{-1}(O)$$

- $\tilde{f}$ is open ($\tilde{f}^{-1}$ is cts)

For $\forall$ open $U \subset X/R$. Hope $\tilde{f}(U)$ open

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$ f is quotient map

$$q^{-1}(U) \text{ open} \Longleftrightarrow f^{-1}(\tilde{f}(U)) \text{ open}$$

$$q^{-1}(U) = f^{-1}(\tilde{f}(U))$$

$x \in q^{-1}(U)$

$\Rightarrow \exists y \in U$, st. $q(x) = y$

$\Rightarrow f(x) = \tilde{f}(q(x)) = \tilde{f}(y)$

$\Rightarrow x \in f^{-1}(\tilde{f}(U))$

$x \in f^{-1}(\tilde{f}(U))$

$\Rightarrow \exists y \in U$ st. $\tilde{f}(y) = f(x)$

$\Rightarrow \tilde{f}(q(x)) = \tilde{f}(y)$

$\overset{\text{inj}}{\Rightarrow} q(x) = y$

$\Rightarrow x \in q^{-1}(U)$

3. (Application of Q2 to prove a quotient space $\simeq$ another topo)

Show that $\mathbb{R}^2/\sim$ is homeomorphic to $(\mathbb{R}^{\geq 0}, \mathcal{T}_{std})$

where $\sim$ is defined as follows:

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if} \quad |x_1| + |y_1| = |x_2| + |y_2|$$

~~Pf~~ : Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^{\geq 0}$

$$(x_1, y_1) \longmapsto |x_1| + |y_1|$$

It is easy to check R defined in Q2 is $\sim$

So all we need is to prove f is quotient map.

- f is surj
- O open $\Rightarrow f^{-1}(O)$ open (f is cts)
- $f^{-1}(O)$ open $\Rightarrow O$ open (f is open)