MATH 2060 Mathematical Analysis II

Tutorial Class 7

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1. Let f, g be continuous function defined on $[a,b]$. Suppose that $f(x) \geq g(x)$ for all $x \in [a,b]$ and $g(x) \neq f(x)$. Show that
$$\int_a^b f > \int_a^b g.$$

2. (a) Define the improper integral $\displaystyle\int_a^\infty f$.

   (b) Let $p \in \mathbb{R}$, show that $\displaystyle\int_1^\infty x^p \, dx$ exists if and only if $p < -1$.

3. (a) Let $f : [a,\infty) \to \mathbb{R}$ be a function such that $f \in R[a,b]$ for all $b > a$. Show that $\displaystyle\int_a^\infty f$ exists if and only if $\forall\, \epsilon > 0$, there exists $K > a$ such that for all $x, y > K$, $\displaystyle\int_x^y f < \epsilon$.

   (b) Let $f, g : [a,\infty) \to \mathbb{R}$ be two function such that $f, g \in \; R[a,b]$ for all $b > a$ and $0 \leq f \leq g$ on $[a,\infty)$.. Show that $\displaystyle\int_a^\infty f$ exists if $\displaystyle\int_a^\infty g$ exists.

4. (a) Show that $\displaystyle\int_1^\infty \frac{\sin x}{x}$ exists .

   (b) Show that $\displaystyle\int_1^\infty \frac{|\sin x|}{x}$ does not exists .

5. (a) Let $a < b$. Suppose $f : (a,b] \to \mathbb{R}$ satisfies $f \in R[c,b]$ for all $c \in (a,b]$. Define the improper integral $\displaystyle\int_a^b f$.

   (b) Let $f : (0,1] \to \mathbb{R}$ be a continuous function. Suppose there exists $C > 0$ and $p > -1$ such that $|f(x)| \leq Cx^p$ for all $x \in (0,1]$. Show that $\displaystyle\int_0^1 f$ exists.