MATH 2060 Mathematical Analysis II
Tutorial Class 2
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1. (a) Dene Riemann integrability of a function.

   (b) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable. Prove that f is bounded.

   (c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if there exists exactly one value A such that

   $$L(f, P) \leq A \leq U(f, P) \text{ for every partition } P \text{ of the interval } [a, b].$$

   (d) Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   Show that f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(f, P) - L(f, P) < \epsilon.$$

2. Show that any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

3. Are the following functions integrable ?

   (a) Let $f : [0, 1] \to \mathbb{R}$,

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{otherwise.} \end{cases}$$

   (b) Let $f : [0, 1] \to \mathbb{R}$.

   $$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for some } n \in \mathbb{N} \\ g(x) & \text{otherwise.} \end{cases}$$

   where $g : [0, 1] \to \mathbb{R}$ is a continuous function.

4. Suppose that a integrable function $f : [a, b] \to \mathbb{R}$ has the property that $f(x) \geq 0$, $\forall x \in [a, b]$. Prove that $\int_b^a f \geq 0$.

5. If $f : [a, b] \to \mathbb{R}$ is a integrable function and $f(x) = C$, $\forall x \in \mathbb{Q} \cap [0, 1]$. Find $\int_b^a f$.