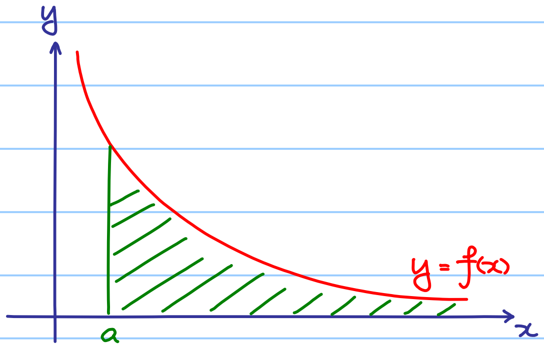
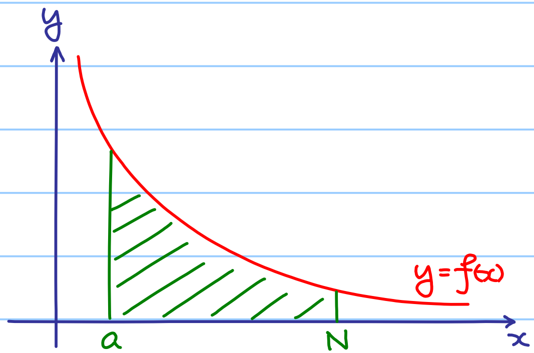
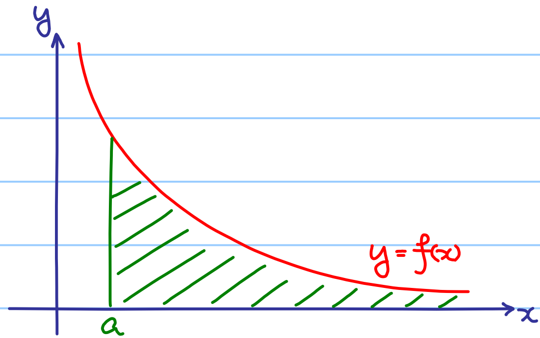
# Improper Integrals :




Question : Find the area of the unbounded region ?

Idea :



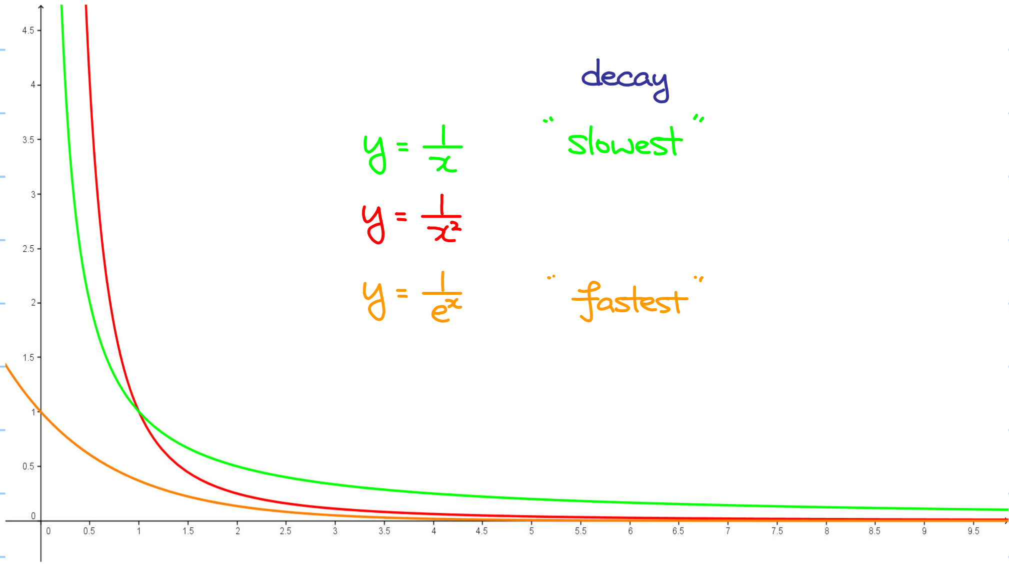




$\int_a^N f(x)\,dx \quad \rightsquigarrow$ Area of the unbounded region

$= \lim_{N \to +\infty} \int_a^N f(x)\,dx$ (if it exists)

We denote it by $\int_a^{+\infty} f(x)\,dx$

e.g.



① $\lim_{N\to+\infty} \int_1^N \frac{1}{x}\,dx = \lim_{N\to+\infty} \left[\ln x\right]_1^N = \lim_{N\to+\infty} \ln N = +\infty$ (i.e. limit does NOT exist)

② $\lim_{N\to+\infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N\to+\infty} \left[-\frac{1}{x}\right]_1^N = \lim_{N\to+\infty} 1 - \frac{1}{N} = 1$

③ $\lim_{N\to+\infty} \int_1^N \frac{1}{e^x}\,dx = \lim_{N\to+\infty} \left[-\frac{1}{e^x}\right]_1^N = \lim_{N\to+\infty} -\frac{1}{e^N} + \frac{1}{e}$

Observation: $\lim_{x\to+\infty} f(x) = 0$ does NOT guarantee $\lim_{N\to+\infty} \int_a^N f(x)\,dx$ exists.

e.g. Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)}\,dx$

Note : $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{N\to+\infty}\int_0^N \frac{1}{(x+1)(3x+2)}\,dx = \lim_{N\to+\infty}\int_0^N \frac{-1}{x+1}+\frac{3}{3x+2}\,dx$$

$$= \lim_{N\to+\infty}\Big[-\ln|x+1| + \ln|3x+2|\Big]_0^N$$

$$= \lim_{N\to+\infty} \ln\left|\frac{3N+2}{N+1}\right| - \ln 2$$

$$= \ln 3 - \ln 2$$

e.g. Find $\int_0^{+\infty} xe^{-2x}\,dx$

$$\lim_{N\to+\infty} \int_0^N xe^{-2x}dx$$

$$= \lim_{N\to+\infty} \int_0^N x\,d\left(-\frac{1}{2}e^{-2x}\right)$$

$$= \lim_{N\to+\infty} \left[-\frac{1}{2}xe^{-2x}\right]_0^N - \int_0^N -\frac{1}{2}e^{-2x}\,dx$$

$$= \lim_{N\to+\infty} \left[-\frac{1}{2}xe^{-2x}\right]_0^N + \left[-\frac{1}{4}e^{-2x}\right]_0^N$$

go to 0 when $N\to+\infty$

$$= \lim_{N\to+\infty} -\frac{1}{2}Ne^{-2N} - \frac{1}{4}e^{-2N} + \frac{1}{4}$$

$$= \frac{1}{4}$$

## Exponential Growth :

### • Simple Interest

Question : How much interest $I$ can one get by putting money into a bank ?

Answer : It depends on :

- how much one put (Principal $P$)
- how long one put (Time $t$ (years))
- which bank one put (Interest rate $r$ )

$\therefore I = P \times t \times r$

$$\text{Amount } A = P + I$$
$$= P \times (1 + t \times r)$$

## • Compound Interest

Nothing, but calculating simple interest repeatly!

e.g. $P = 10000$, $r = 5\%$, compound monthly (12 times per year),

deposit for 2 years.

$$A = 10000 \times \left(1 + \frac{5\%}{12}\right)^{12 \times 2}$$

e.g. $P$, $r$, compound monthly $k$ times per year,

deposit for $T$ years.

$$A = P\left(1 + \frac{r}{k}\right)^{kT}$$

Compound continuously $\Rightarrow$ let $k \to +\infty$

$$\text{Amount} = \lim_{k\to+\infty} P\left(1+\frac{r}{k}\right)^{kT}$$

$$= \lim_{k\to+\infty} P \times \left[\left(1+\frac{1}{\left(\frac{k}{r}\right)}\right)^{\frac{k}{r}}\right]^{rT}$$

$$= Pe^{rT}$$

(which is called exponential growth)

e.g. Suppose \$1000 is invested at an annual interest rate of 7%.

Compute the amount after 10 years if the interest is compounded

a) Annually $(k=1)$ Ans: $A = 1000 \times \left(1+\frac{7\%}{1}\right)^{1\times 10} \approx 1967.15$

$\wedge$

b) Quarterly $(k=4)$ Ans: $A = 1000 \times \left(1+\frac{7\%}{4}\right)^{4\times 10} \approx 2001.60$

$\wedge$

c) Monthly $(k=12)$ Ans: $A = 1000 \times \left(1+\frac{7\%}{12}\right)^{12\times 10} \approx 2009.66$

$\wedge$

d) Continuously $(k\to\infty)$ Ans: $A = 1000 \times e^{7\%\times 10} \approx 2013.75$

## Exponential Growth / Decay :

In general ,

$Q_0$ = value at the beginning $(t=0)$

$r$ = growth $(r>0)$ / decay $(r<0)$ rate

$t$ = time (years)

$Q(t)$ = value at time $t$

$= Q_0 e^{rt}$

e.g. Depreciation (Depreciate continuously)

Value of a machine now = 20000

Rate of depreciation = 40% / year

Value of a machine after 5 years $= 20000 e^{-0.4 \times 5}$

$\approx 2706.71$

Application :

Quanities modelled in terms of exponential growth / decay :

- population growth of bacteria
- decay of radioactive substances
- concentration of drug in a patient's bloodstream
- and etc

## Relative Rates of Change

Think:

$Q(t)$ : salary after working for $t$.

$Q'(t)$ : rate of change of salary.

For example, $Q'(t) > 0$, that's good!

But, how good?

You have to compare it with the salary at that moment!

i.e. $\frac{Q'(t)}{Q(t)}$ and it is called its relative rate of change of $Q$.

In general, relative rate of change of $Q(x)$ (w.r.t. $x$) $= \frac{Q'(x)}{Q(x)}$

$$= \frac{d}{dx} \ln Q(x)$$

## Revisit of Exponential Growth / Decay

Suppose $Q(x)$ follows exponential growth / decay , i.e. $Q(x) = Q_0 e^{rx}$

Then relative rate of change of $Q(x)$

$$= \frac{d}{dx} \ln Q(x)$$

$$= \frac{d}{dx} \ln Q_0 e^{rx}$$

$$= \frac{d}{dx} \ln Q_0 + rx$$

$$= r$$

$\therefore$ If $Q(x)$ follows exponential growth / decay ,

then relative rate of change of $Q(x) = r =$ constant

Note , $Q'(x) = r\,Q(x)$ ,

e.g. suppose $r = -2\%$ , $Q'(x) = -0.02\,Q(x)$ ,

it means the rate of change of $Q(x)$ is $-2\%$ .

## Additional Applications of Definite Integration

e.g.

A new clinic has just established. Past statistics shows that fraction of patients who will still be receiving treatment at the clinic $t$ months after their initial visit is given by the function $f(t) = e^{-t/10}$. The clinic initially accepts 500 patients plans to accept new patients at the constant rate $g(t) = 200$. Approximately, how many will be receiving treatment at the clinic 6 months from now?

- $f(t)$ should be realized at the probability of a patient is still at the clinic after $t$ months
(Exponential decay)

(I) First 500 patients :

Q: How many of them will remain at the clinic 6 months from now ?

A: Simply $\underset{\text{Number}}{500} \times \underset{\text{Probability}}{f(6)}$

(II) Patients come later on :


$\Delta t = t_{i+1} - t_i$


Number of patients receiving from $t_i$ to $t_{i+1}$ = $200(t_{i+1}-t_i) = 200\Delta t$

Number of them will remain at the clinic 6 months from now ?

Summing up over $0 \leq t \leq 6$ :

$$\lim_{n\to\infty} \sum_{i=1}^{n} 200 f(6-t_i)\Delta t = \int_0^6 200 f(6-t)\,dt$$

$\therefore$ Number of patients after 6 months

$$= \underbrace{500 f(6)}_{(\mathrm{I})} + \underbrace{\int_0^6 200 f(6-t)\,dt}_{(\mathrm{II})}$$

$$= 500 e^{-6/10} + \int_0^6 200 e^{-(6-t)/10}\,dt$$

$$= 500 e^{-6/10} + 200 e^{-6/10} \int_0^6 e^{t/10}\,dt$$

$$= 500 e^{-6/10} + 200 e^{-6/10} \left[\frac{1}{10} e^{t/10}\right]_0^6$$

$$= 500 e^{-6/10} + 200 e^{-6/10} \left[\frac{1}{10} \left(e^{6/10} - 1\right)\right]$$

$$\approx 283$$

Generalization :

$P(t)$ : value of a quanity

$P_0 = P(0)$ : initial value

$R(t)$ : renewal rate

$S(t)$ : survival function

Then $P(T) = P_0 S(T) + \int_0^T R(t) S(T-t)\, dt$

e.g. It is estimated that $t$ years from now, a neclear plant will be producing nuclear waste at the rate 400 pounds / year. The waste decays exponentially at the rate of 2% per year. Assume that there is no radioactive waste at the beginning. How much radioactive waste will eventually accumulate?

$P(t)$ : Amount of radioactive waste at time $t$ (years)

$P_0 = 0$ : Initial amount of radioactive waste

$R(t)$ : $400$

$S(t)$ : $e^{-0.02t}$

Radioactive waste will eventually accumulate

$$= \lim_{T\to+\infty} \int_0^T 400\, e^{-0.02(T-t)}\, dt$$

Ex :

$$= \lim_{T\to+\infty} 20000(1-e^{-0.02T})$$

$= 20000$ pounds.