Integration of Exponential Functions :

Recall : $\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$

In general : $\int a^x\,dx = ?$ for $a > 0$

Recall : $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$$\therefore \int a^x\,dx = \int e^{(\ln a)x}\,dx$$

$$= \frac{1}{\ln a} e^{(\ln a)x} + C$$

$$= \frac{a^x}{\ln a} + C$$

OR : Recall that $\frac{d}{dx} a^x = a^x \ln a$

so $\frac{d}{dx} \frac{a^x}{\ln a} = a^x$, and $\int a^x\,dx = \frac{a^x}{\ln a} + C$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Ex : $\frac{d}{dx} x\ln x - x$

Ans : $\ln x$!

Therefore , $\int \ln x \, dx = x\ln x - x + C$

Problem : How do we know $\frac{d}{dx} x\ln x - x = \ln x$ in advance ?

(Make a guess of antiderivative of $\ln x$ directly)

Any direct way to find an antiderivative of $\ln x$? (Yes, later !)

e.g. (Constant issue)

$$\int (x+1)^2\,dx \qquad \text{let } u = x+1$$
$$= \int u^2\,du \qquad du = dx$$
$$= \frac{1}{3}u^3 + C$$
$$= \frac{1}{3}(x+1)^3 + C$$
$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$$

$$\int (x+1)^2\,dx$$
$$= \int x^2+2x+1\,dx$$
$$= \frac{1}{3}x^3 + x^2 + x + C$$

seems to be different !

Ans : This C is NOT that C !

Integration by Parts

Recall : Let $u(x)$ and $v(x)$ be differentiable functions .

Product rule : $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both sides with respect to x :

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

OR : $\int u\,dv = uv - \int v\,du$

Integration by Parts : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

e.g. $\int x^2 \ln x \, dx = \int (\ln x) \, x^2 \, dx$

$= \int (\ln x) \frac{d}{dx}\left(\frac{x^3}{3}\right) dx$ (Now, $u = \ln x$, $v = \frac{x^3}{3}$)

$= \int \ln x \, d \frac{x^3}{3}$

$= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \, d(\ln x)$

$= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} dx$

$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$ (Verify the answer by differentiation!)

e.g. $\int xe^x dx$

Note : $\frac{d}{dx}e^x = e^x$

$e^x dx = de^x$

$\int x e^x dx$

$= \int x\, de^x$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

$= e^x(x-1) + C$

Now, $u = x$, $v = e^x$

e.g. $\int x^2 e^x dx$

$= \int x^2 de^x$

$= x^2 e^x - \int e^x dx^2$

$= x^2 e^x - \underbrace{\int 2xe^x dx}$

Ex: : Apply Integration by parts again!

Ans : $e^x(x^2-2x+2)+C$

Challenge : $\int x^n e^x dx = ?$

Ans : $e^x[x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \cdots + (-1)^n n!] + C$

Question : How to make a guess of u(x) and v(x) ?

Integration by Parts : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

e.g. $\int x^2 \ln x \, dx = \int (\ln x) \, x^2 \, dx$

$= \int (\ln x) \frac{d}{dx}\left(\frac{x^3}{3}\right) dx$

Realize the integrand as a product of parts and make a guess of u(x) and v(x) such that one part can be realized as a function $u(x)$, another part is $v'(x)$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Using Integration by part :

$$\int \ln x \, dx \qquad u = \ln x \qquad v = x$$

$$= x \ln x - \int x \, d \ln x$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Ex: $\int \log_a x \, dx = ?$

Hints: $\log_a x = \frac{\ln x}{\ln a}$

$$\int \log_a x \, dx = \frac{1}{\ln a} \int \ln x \, dx$$

$$= \frac{1}{\ln a} (x \ln x - x + C)$$

$$= x \frac{\ln x}{\ln a} - \frac{x}{\ln a} + \frac{C}{\ln a}$$

$$= x \log_a x - \frac{x}{\ln a} + C'$$

$C' = \frac{C}{\ln a}$ just a constant!

Sequences of Real Numbers

e.g. $a_1 = 2$, $a_2 = \pi$, $a_3 = 1$, $\cdots$

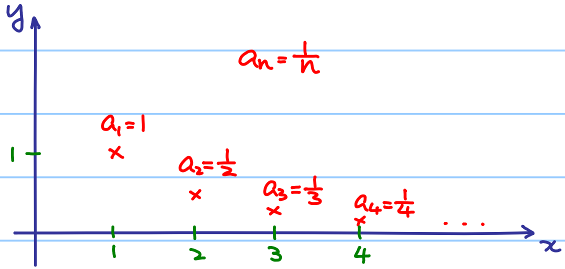
OR write as $\{2, \pi, 1, \cdots\}$ (No pattern)

e.g. Sequences having patterns.

$a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $\cdots$ in general, $a_n = 2^{n-1}$

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $\cdots$ in general, $a_n = \frac{1}{n}$

$a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\cdots$ in general, $a_n = (-1)^n$



Any observation?

When n is getting larger and larger, a_n is getting closer and closer to 0.

Limits of Sequences

Informal definition:

Let a_n be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$, then we say L is the limit of the sequence a_n and we denote it by

$$\lim_{n \to \infty} a_n = L.$$

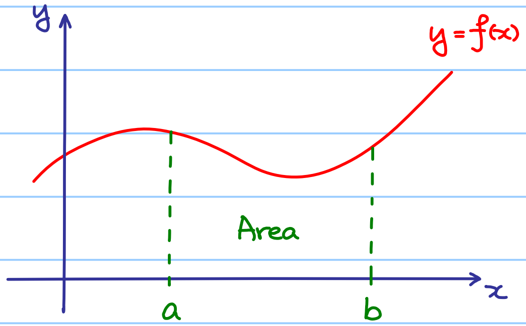
e.g. $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\lim_{n\to\infty} (-1)^n$ does NOT exist.

$\lim_{n\to\infty} 2^{n-1}$ does NOT exist.

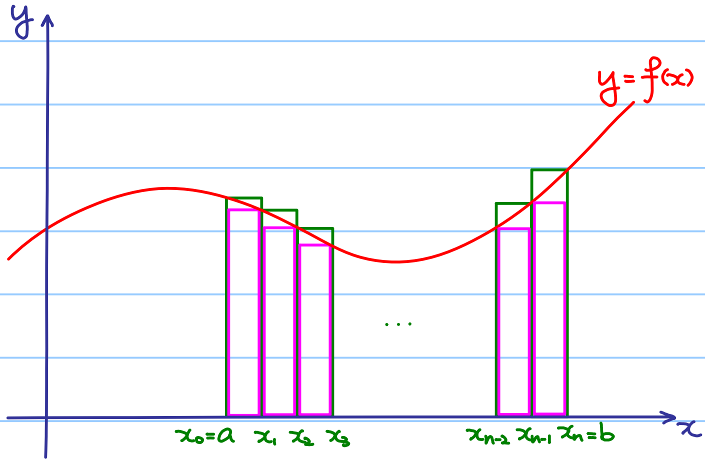
Definite Integration

Goal : Find the area of the region under the curve $y = f(x)$ over an interval $[a, b]$.



Riemann Sum

Area as the limit of a sum



Subdivide $[a, b]$ into n equal subintervals, $x_i - x_{i-1} = \Delta x$, $i = 1, 2, 3, \cdots, n$

$$\text{Upper sum} = \max_{x_0 \le \xi_1 \le x_1} f(\xi_1)\,\Delta x + \max_{x_1 \le \xi_2 \le x_2} f(\xi_2)\,\Delta x + \cdots + \max_{x_{n-1} \le \xi_n \le x_n} f(\xi_n)\,\Delta x$$

$$U_n = \sum_{i=1}^{n} \max_{x_{i-1} \le \xi_i \le x_i} f(\xi_i)\,\Delta x$$

$$\text{Lower sum} = \min_{x_0 \le \xi_1 \le x_1} f(\xi_1)\,\Delta x + \min_{x_1 \le \xi_2 \le x_2} f(\xi_2)\,\Delta x + \cdots + \min_{x_{n-1} \le \xi_n \le x_n} f(\xi_n)\,\Delta x$$

$$L_n = \sum_{i=1}^{n} \min_{x_{i-1} \le \xi_i \le x_i} f(\xi_i)\,\Delta x$$

Note : $L_n \le \text{Area} \le U_n$

Rough idea : $n \to \infty$, more rectangles , better approximation !

If $\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} U_n = A$, we define the area to A . —— (*)

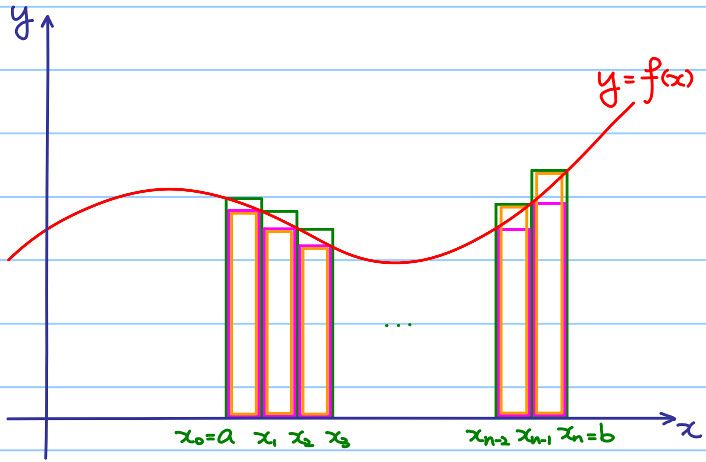
Remark :

1) If the area is defined, we denote it by $\int_a^b f(x)\,dx$.

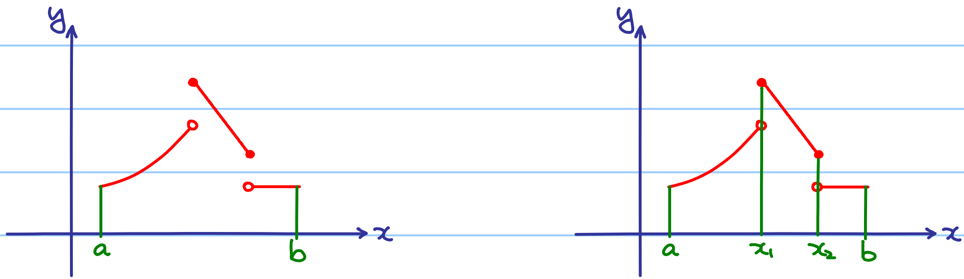
2) If $f(x)$ is a continuous function, $\int_a^b f(x)\,dx$ is well-defined for any $a \leq b$.

3) Let $a_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(a+(b-a)\frac{i}{n}\right)\cdot\frac{b-a}{n}$, then $L_n \leq a_n \leq U_n$.

Now, we know $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = A$, so $\lim_{n\to\infty} a_n = A$

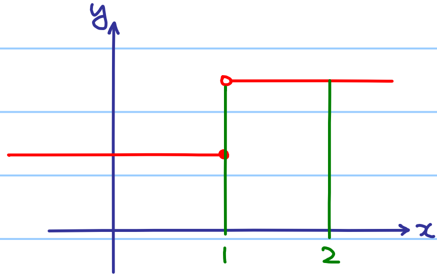


4) If f is a piecewise continuous on $[a,b]$, i.e. discontinuous only at finitely many points, then $\int_a^b f(x)\,dx$ is defined as the following :



$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{b} f(x)\,dx$$

e.g. Let $f(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$



$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= 1 \times 1 + 2 \times 1$$

$$= 3$$

Note: width of a point $= 0$

Rules for Definite Integrals :

Let $f(x)$, $g(x)$ be continuous (or piecewise continuous) functions.

Suppose $a \leq b$.

1) If k is a constant, $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

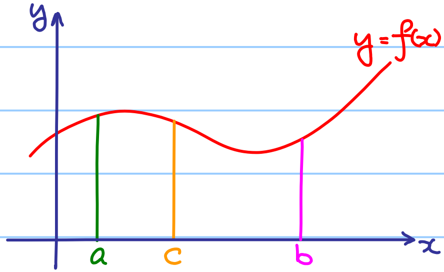
2) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

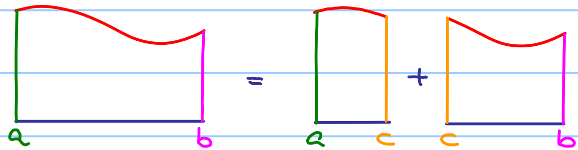
3) $\int_a^a f(x)\,dx = 0$

4) $\int_b^a f(x)\,dx$ is defined to be $-\int_a^b f(x)\,dx$ (reverse direction)

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any c (subdivision)

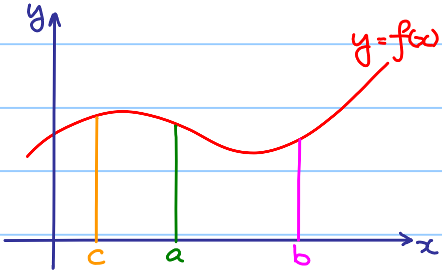
If $a \leq c \leq b$,

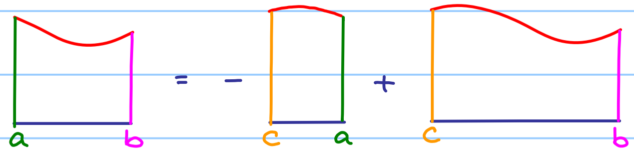




$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $c < a \leq b$,





$$\int_a^b f(x)dx = \underbrace{\int_a^c f(x)dx}_{\substack{\parallel \\ -\int_c^a f(x)dx}} + \int_c^b f(x)dx$$

Ex: Think why (5) is true if $a \leq b < c$!

These properties are followed the definition (*).

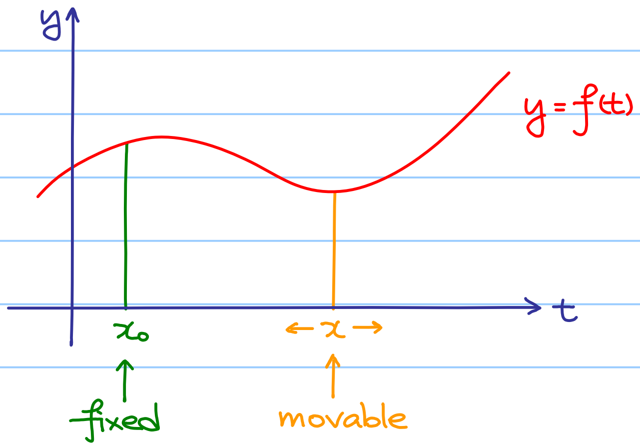
Computation of area :

NOT rely on the above limit , but Fundamental Theorem of Calculus !

Fundamental Theorem of Calculus :

Preparation :

Let $f(t)$ be a continuous function.



1) $\int_{x_0}^{x} f(t)dt$ is well defined for all $x \in \mathbb{R}$

2) What is a function? Roughly speaking, input x, output y.

Now, construct a new function $F(x)$ defined by

$F(x)$ = Area under the curve $y=f(t)$ over $[x_0, x]$

$= \int_{x_0}^{x} f(t)dt$

3) How about choosing another fixed point?

Let $\tilde{F}(x) = \int_{x_1}^{x} f(t)dt$, what is the difference between $F(x)$ and $\tilde{F}(x)$?

In fact,

$$F(x) - \tilde{F}(x) = \int_{x_0}^{x} f(t)dt - \int_{x_1}^{x} f(t)dt$$

$$= \int_{x_0}^{x} f(t)dt + \int_{x}^{x_1} f(t)dt$$

$$= \int_{x_0}^{x_1} f(t)dt \quad \text{which is a constant.}$$

Fundamental Theorem of Calculus :

Let $f(t)$ be a continuous function, x_0 be a fixed point.

Suppose $F(x)$ is a function defined by

$$F(x) = \int_{x_0}^{x} f(t)\,dt ,$$

then $F(x)$ is a differentiable function and $F'(x) = f(x)$.

(i.e. $F(x)$ is an antiderivative of $f(x)$.)

1) Direct consequence : $\int_a^b f(x)\,dx = \int_{x_0}^{b} f(x)\,dx - \int_{x_0}^{a} f(x)\,dx$

$$= F(b) - F(a)$$

i.e. if we know how to compute antiderivative of $f(x)$,

then we know how to find $\int_a^b f(t)\,dt$.

2) Wait! Antiderivative of $f(x)$ is NOT unique, but unique up to a constant.

Which one should we pick?

If $\tilde{F}(x) = \int_{x_1}^{x} f(t)\,dt$, then $\tilde{F}(x)$ is another antiderivative of $f(x)$.

In fact, it is NOT surprising, we know $F(x) - \tilde{F}(x)$ is a constant.

Also, $\int_a^b f(x)\,dx = \int_{x_1}^{b} f(x)\,dx - \int_{x_1}^{a} f(x)\,dx$

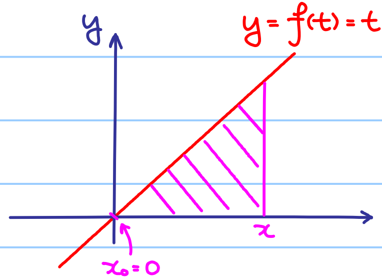
$= \tilde{F}(b) - \tilde{F}(a)$

Therefore, we can pick anyone!

e.g. (Verification of Fundamental Theorem of Calculus)

$f(t) = t \quad , \; x_0 = 0$

$f(x) = x$



$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

$= \text{Area of the shaded triangle}$

$= \frac{1}{2}x^2$

Note : We have $F'(x) = f(x)$.

e.g. $f(x) = x+1$

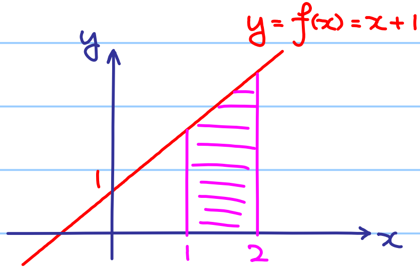
Antiderivative of $f(x) = \int x+1\,dx = \frac{x^2}{2} + x + C$

Choose $C=0$, let $F(x) = \frac{x^2}{2} + x$

Area of the shaded region $= \int_1^2 f(x)\,dx = F(2) - F(1)$

$= 4 - \frac{3}{2}$

$= \frac{5}{2}$



What we write :

$\int_1^2 f(x)\,dx = \left[\frac{x^2}{2} + x\right]_1^2$

$= \underbrace{\left(\frac{2^2}{2} + 2\right)}_{F(2)} - \underbrace{\left(\frac{1^2}{2} + 1\right)}_{F(1)} = 4 - \frac{3}{2} = \frac{5}{2}$

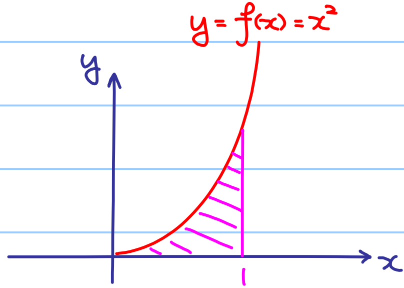
e.g. $f(x) = x^2$

Area of the shaded region $= \int_0^1 f(x)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^1$$

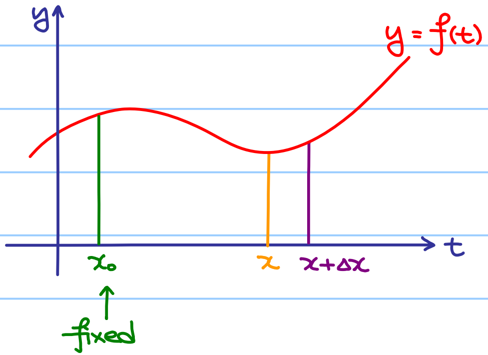
$$= \left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right)$$

$$= \frac{1}{3}$$



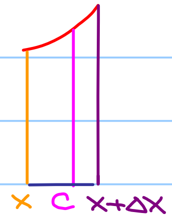
Sketch of the proof Fundamental Theorem of Calculus

Claim: If $F(x) = \int_{x_0}^{x} f(t)\,dt$, $\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x)$, i.e. $F'(x) = f(x)$



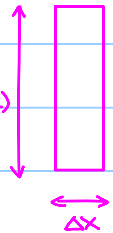
$F(x+\Delta x) - F(x)$

$=$ Area of



By continuity of f, there exists $c \in (x, x+\Delta x)$ such that

$= f(c)$



$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(c)$$

$$= \lim_{c \to x} f(c)$$ (As Δx tends to 0, c tends to x)

$$= f(x)$$ (By continunity of f)