Algebraic Rules For Indefinite Integration

1) $\int k f(x)\,dx = k\int f(x)\,dx$

2) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Note : 1) $\frac{d}{dx}\left(\int k f(x)\,dx\right) = \frac{d}{dx}\left(k\int f(x)\,dx\right) = k f(x)$

i.e. $\int k f(x)\,dx$ and $k\int f(x)\,dx$ differ by a constant.

but it is absorbed by $\int$.

2) $\frac{d}{dx}\left(\int f(x) \pm g(x)\,dx\right) = \frac{d}{dx}\left(\int f(x)\,dx \pm \int g(x)\,dx\right) = f(x) \pm g(x)$.

e.g. $\int 2x^5 - 3x^2 + 7x + 5 \, dx$

$\int dx$ means $\int 1 \, dx$

$= 2\int x^5 dx - 3\int x^2 dx + 7\int x \, dx + 5\int dx$

$\int$ still there, No need to add $+C$!

$= 2\left(\frac{x^6}{6}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

e.g. $\int \frac{x^3-5}{x}\,dx$

$= \int x^2 - \frac{5}{x}\,dx$

$= \frac{x^3}{3} - 5\ln|x| + C$

e.g. Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

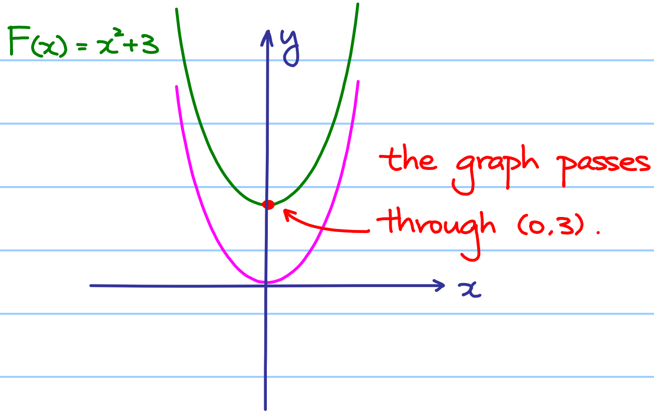
$F'(x) = 2x$

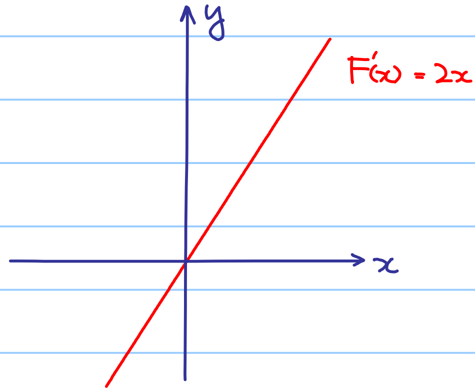
$F(x) = \int 2x \, dx$

$= x^2 + C$

$F(0) = 0^2 + C = 3 \Rightarrow C = 3$

$\therefore F(x) = x^2 + 3$


$F(x) = x^2+3$
y
x
the graph passes
through (0,3).


y
x
$F'(x) = 2x$

Integration by Substitution

Question : $\int (2x+1)^{2015} dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Integration by Substitution : $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

OR : $\int f(x)\frac{du}{dx}\,dx = \int f(u)\,du$

proof : $\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = f(u(x))\,u'(x)$

$$\frac{d}{dx}\int f(u)\,du = \frac{d}{du}\int f(u)\,du \cdot \frac{du}{dx}$$ (Chain Rule)

$$= f(u(x)) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = \frac{d}{dx}\int f(u)\,du$$

$$\therefore \int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

e.g. $\int (2x+1)^{2015}\,dx = ?$

Let $u(x) = 2x+1 \qquad u'(x) = 2$

$f(u) = u^{2015} \qquad f(u(x)) = (2x+1)^{2015}$

$$\int (2x+1)^{2015}\,dx = \frac{1}{2}\int \underbrace{(2x+1)^{2015}}_{f(u(x))} \cdot \underbrace{2}_{u'(x)}\,dx = \frac{1}{2}\int \underbrace{u^{2015}}_{f(u)}\,du$$

$$= \frac{1}{4032}u^{2016} + C = \frac{1}{4032}(2x+1)^{2016} + C$$

But, usually we write,

$\int (2x+1)^{2015}\,dx$

$= \int u^{2015}\,\frac{1}{2}du$

$= \frac{1}{4032}u^{2016} + C$

$= \frac{1}{4032}(2x+1)^{2016} + C$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

e.g. $\int e^{ax}\,dx$

$= \int e^{u} \cdot \frac{1}{a}\,du$

$= \frac{1}{a} e^{u} + C$

$= \frac{1}{a} e^{ax} + C$

Let $u = ax$

$\frac{du}{dx} = a$

$dx = \frac{1}{a}\,du$

e.g. $\int 6x(4x^2+3)^7\,dx$

$= \int 6(4x^2+3)^7\,x\,dx$

$= \int 6u^7 \frac{1}{8}\,dx$

$= \frac{6}{8} \cdot \frac{1}{8} u^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

Let $u = 4x^2+3$

$\frac{du}{dx} = 8x$

$x\,dx = \frac{1}{8}\,du$

e.g. $\int \frac{(\ln x)^2}{x}\,dx \quad , \quad x>0$

$\int \frac{(\ln x)^2}{x}\,dx$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\frac{1}{x}\,dx = du$

$= \int u^2\,du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}(\ln x)^3 + C$

Question : How to make a guess of $u(x)$?

Integration by Substitution : $\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$

e.g. $\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \cdot \frac{1}{x} dx$ Let $u = \ln x$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$

Ex : 1) Show that $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$. Hint : Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4} dx$ Hint : Let $u = x^4$ Ans : $\frac{1}{4} e^{x^4} + C$

b) $\int 6x\sqrt{x^2+3}\, dx$ Hint : Let $u = x^2+3$ Ans : $2(x^2+3)^{\frac{3}{2}} + C$

Integration of Rational Functions :

- $\int \frac{p(x)}{ax+b}\,dx$

$$\begin{array}{r|l} & q(x) \\ \hline ax+b & p(x) \\ & \vdots \\ \hline & R \end{array}$$

By long division, $p(x) = (ax+b)\,q(x) + R$

$$\frac{p(x)}{ax+b} = q(x) + \frac{R}{ax+b}$$

Then $\int \frac{p(x)}{ax+b}\,dx = \int q(x) + \frac{R}{ax+b}\,dx$

We know how to integrate !

e.g. $\displaystyle\int \frac{x^2+3x+5}{x+1}\,dx$

$$
\begin{array}{r}
x+2 \\
x+1\,\overline{)\,x^2+3x+5} \\
\underline{x^2+x} \\
2x+5 \\
\underline{2x+2} \\
3
\end{array}
$$

$= \displaystyle\int x+2+\frac{3}{x+1}\,dx$

$= \dfrac{x^2}{2}+2x+3\ln|x+1|+C$

$\therefore\ x^2+3x+5 = (x+1)(x+2)+3$

$\dfrac{x^2+3x+5}{x+1} = x+2+\dfrac{3}{x+1}$

Ex : Evaluate $\displaystyle\int \frac{6x^2-5x+1}{3x-2}\,dx$

Ans : $x^2-\dfrac{x}{3}+\dfrac{1}{9}\ln|3x-2|+C$

- $\displaystyle\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx$

Express $\dfrac{ax+b}{(r_1x+s_1)(r_2x+s_2)}$ into the form $\dfrac{A}{r_1x+s_1}+\dfrac{B}{r_2x+s_2}$.

Then $\displaystyle\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{r_1x+s_1}+\frac{B}{r_2x+s_2}\,dx$

We know how to integrate!

e.g. $\int \frac{5x-7}{x^2-2x-3}\,dx$

Note : $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$\Rightarrow \quad 5x-7 \equiv A(x+1) + B(x-3)$

$\Rightarrow \quad A = 3 \;,\; B = 2 \,.$

$$\int \frac{5x-7}{x^2-2x-3}\,dx = \int \frac{3}{x-3} + \frac{2}{x+1}\,dx = 3\ln|x-3| + 2\ln|x+1| + C$$

Ex : Evaluate $\int \frac{40}{x(200-x)}\,dx$

Ans : $\frac{1}{5}(\ln|x| - \ln|200-x|) + C = \frac{1}{5}\ln\left|\frac{x}{200-x}\right| + C$

- $\int \frac{ax+b}{(px+q)^2}\,dx$

Express $\frac{ax+b}{(px+q)^2}$ into the form $\frac{A}{(px+q)^2} + \frac{B}{px+q}$

Then $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{(px+q)^2} + \frac{B}{px+q}\,dx$

We know how to integrate!

e.g. $\displaystyle\int \frac{2x-1}{(x-2)^2}\,dx$

Suppose $\displaystyle\frac{2x-1}{(x-2)^2} \equiv \frac{A}{(x-2)^2} + \frac{B}{x-2}$

$\Rightarrow 2x-1 \equiv A + B(x-2)$

$\Rightarrow A=3, \ B=2$

$$\int \frac{2x-1}{(x-2)^2}\,dx = \int \frac{3}{(x-2)^2} + \frac{2}{x-2}\,dx = \frac{-3}{x-2} + 2\ln|x-2| + C$$

Ex: Evaluate $\displaystyle\int \frac{4x+2}{(2x-1)^2}\,dx$

Ans: $\displaystyle\frac{-2}{2x-1} + \ln|2x-1| + C$

Remarks :

1) If $\deg p(x) > 1$, $\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = ?$

Hint : Long division .

$$\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int q(x) + \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx$$

reduced to pervious case !

2) If ax^2+bx+c cannot be factorized as a product of two linear factors (i.e. $b^2-4ac<0$) , then $\int \frac{1}{ax^2+bx+c}\,dx = ?$

Unfortunately, we cannot cover this case as it involves trigonometric functions !