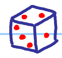
Probability

We all learned simple probability before !

e.g. Roll a die $\rightsquigarrow$  $P(2) = \frac{1}{6}$

Toss a coin $\rightsquigarrow$  $P(\text{head}) = \frac{1}{2}$

Goal : Make things formal

Discrete $\longrightarrow$ Continuous

Terminologies :

1) Possible outcome : possible result of a single experiment
2) Sample space : collection of all possible outcomes
3) Events : Any collection of possible outcomes

e.g. A box containing 3 balls , marked 1 , 2 and 3

Randomly select one .

1) Possible outcome $1, 2, 3$
2) Sample space $S = \{1, 2, 3\}$
3) Events $\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$\uparrow$ empty

Event A = the number marked on the selected ball is less than 3 $= \{1, 2\}$

Event B = the number marked on the selected ball is $0 = \phi$

4) Random variable : A function X that assigns a real number to each possible outcome

(i.e. $X : S \longrightarrow \mathbb{R}$)

Refer to the pervious example :

(i) Let X be the number marked on the selected ball.

value assigned

$X(1) = 1$, $X(2) = 2$, $X(3) = 3$

possible outcome

(ii) Let X be the square of the number marked on the selected ball.

value assigned

$X(1) = 1$, $X(2) = 4$, $X(3) = 9$

possible outcome

If the image of X is finite or countably infinite, then X is called a discrete random variable, if the image of X is uncountably infinite, then X is called a continuous random variable.

e.g. Toss a coin 3 times

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ 8 possible outcomes

Let X be the number of times a head comes out.

$X(HHH) = 3$, $X(HHT) = 2$, ...

(image of $X = \{0,1,2,3\}$ which is finite, so X is discrete.)

We can use $X=2$ to denote all possible outcomes with value 2

$X=2 \;=\; \{HHT, HTH, THH\}$

so $X=2$ can be regarded as an event.

$X=2$ is the event of getting 2 heads.

In general, $X=x$ is the event of getting x heads.

e.g. Toss a coin until a head comes out.

$S = \{H, TH, TTH, TTTH, \cdots\}$

Let X be the number of trials.

$X(H) = 1$, $X(TH) = 2$, $X(TTH) = 3$, $\cdots$

(image of $X = \{1, 2, 3, \cdots\}$ which is countably infinite, so X is discrete.)

e.g. There are four students A, B, C, D with heights 160 cm, 165 cm, 170 cm, 180 cm respectively.

Randomly pick two of them.

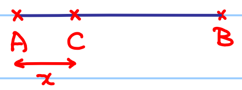
$S = \{AB, AC, AD, BC, BD, CD\}$

Let X be the sum of the heights of the two selected students

$X(AB) = 160 + 165 = 325$, $\cdots$

(image of X is finite, so X is discrete.)

e.g. Suppose AB is a rope of length 10 cm and AB is cut into two pieces.



$S = (0, 10)$

Let X be the square of length of AC^2, so $X(x) = x^2$.

(image of $X = (0, 100)$ which is uncountably infinite, so X is continuous.)

Probability Distribution (Discrete random variable)

S : sample space

X : discrete random variable

A probability distribution (or probability density function pdf) is a function p such that

$$p(x) = P(X=x)$$
$$= \text{Probability of the event } X=x$$

e.g. Toss a coin 3 times

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let X be the number of times a head comes out.

$p(0) = P(X=0) = \frac{1}{8}$

$p(1) = P(X=1) = \frac{3}{8}$

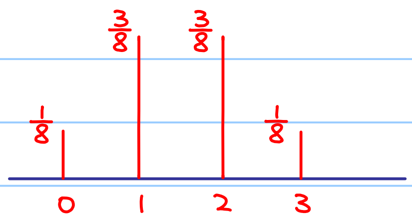
$p(2) = P(X=2) = \frac{3}{8}$

$p(3) = P(X=3) = \frac{1}{8}$

$p(x) = 0$ if $x \neq 0, 1, 2, 3$

Note : $\displaystyle\sum_{x;\, p(x)\neq 0} p(x) = 1$ (summing over x with $p(x) \neq 0$)

Graphical representation :



e.g. Toss a coin until a head comes out.

$S = \{H, TH, TTH, TTTH, \cdots\}$

Let X be the number of trials.

$p(1) = P(X=1) = \frac{1}{2}$

$p(2) = P(X=2) = \frac{1}{4}$

$p(3) = P(X=3) = \frac{1}{8}$

$p(4) = P(X=4) = \frac{1}{16}$

$\vdots$

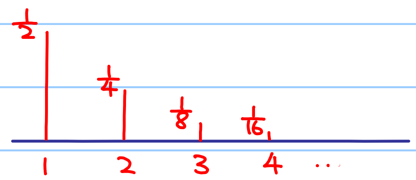
$$\therefore \quad p(x) = \begin{cases} \frac{1}{2^{x+1}} & \text{if } x \text{ is a nonnegative integer} \\ 0 & \text{otherwise} \end{cases}$$

Again $\sum_{x;P(x)\neq 0} P(x) = P(0) + P(1) + P(2) + P(3) + \cdots$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$= 1$

Graphical representation :



Probability Distribution (Discrete random variable)

S : sample space

X : discrete random variable

A probability distribution (or probability density function pdf) is a function p such that

1) $p(x) = P(X=x)$ = Probability of the event $X=x$

2) $p(x) \geq 0$

3) $\sum_{x;\, p(x)\neq 0} p(x) = 1$

Expected Value of a Discrete Random Variable

X : discrete random variable

Expected Value of $X = E(X)$

$$= \sum_{x;\, p(x)\neq 0} x\, p(x)$$

We also call it mean and denote it by μ.

e.g. Roll a die

Sample space $= S = \{1,2,3,4,5,6\}$

Let X be the random variable that denotes the number facing up.

$$p(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1,2,3,4,5,6 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E(X) &= \sum_{x;\, p(x)\neq 0} x\, p(x) \\ &= 1\cdot p(1) + 2\cdot p(2) + \cdots + 6\cdot p(6) \\ &= 3.5 \end{aligned}$$

Q : What is the meaning of this 3.5 ?

Think : If we roll a die for 600 times,

we expect (but NOT necessary to be true) that each number occurs $\frac{1}{6}\times 600 = 100$ times , so the average of the result is

$$\frac{100\times 1 + 100\times 2 + \cdots + 100\times 6}{600} = \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \cdots + \frac{1}{6}\times 6 = 3.5$$

Variance of a Discrete Random Variable

X : discrete random variable

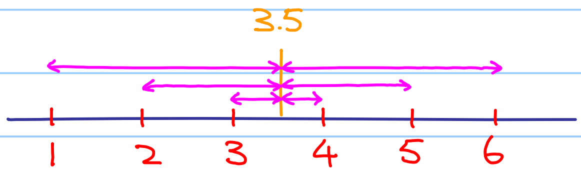
Variance of X = $Var(X)$

$$= \sum_{x: p(x) \neq 0} (x-\mu)^2 p(x)$$

e.g. Roll a die

$$Var(X) = (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6}.$$

$$= \frac{35}{12}$$



Variance provides a measure of the tendency of the values of X to spread out of the expected value $E(X)$.

standard derivation $\sigma := \sqrt{Var(X)}$

Recall : In statistics, $\sigma = \sqrt{\dfrac{\Sigma f_i (x_i - \mu)^2}{n}}$

$$= \sqrt{\Sigma p_i (x_i - \mu)^2}$$

same formula !

x_i : values

f_i : frequency

n : total number

$p_i = \frac{f_i}{n}$

More examples :

Geometric Random Variables

Experiment : Only two possible result

		probability
success	(S)	p
failure	(F)	$1-p$

Continue to try until one success trial .

Sample space = $\{S, FS, FFS, FFFS, \dots\}$

Let X be the random variable that counts the number of trials.

The event $X=x$ is just the event $\underbrace{FF\cdots F}_{x-1 \text{ times}}\, S$ ← x-th trial

$$p(x) = \begin{cases} (1-p)^{x-1}p & \text{if } x \text{ is a positive integer} \\ 0 & \text{otherwise} \end{cases}$$

① $$\sum_{x;\,p(x)\neq 0} p(x) = \sum_{n=1}^{\infty} (1-p)^{n-1}p$$
$$= p\left[1 + (1-p) + (1-p)^2 + \cdots\right]$$
$$= p \cdot \frac{1}{1-(1-p)}$$
$$= 1$$

② $$E(X) = \sum_{x;\,p(x)\neq 0} x\,p(x)$$
$$= \sum_{n=1}^{\infty} n\,p(n)$$
$$= \sum_{n=1}^{\infty} np(1-p)^{n-1}$$
$$= p\sum_{n=1}^{\infty} n(1-p)^{n-1}$$
$$= \frac{1}{p}$$

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$
$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$
put $x = 1-p$, $\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2}$

③ $\mathrm{Var}(X) = \sum_{x:p(x)\neq 0} (x-\mu)^2 p(x)$

$= \sum_{n=1}^{\infty} \left(n-\frac{1}{p}\right)^2 \cdot p(1-p)^{n-1}$

$= p\sum_{n=1}^{\infty} n^2(1-p)^{n-1} - 2\sum_{n=1}^{\infty} n(1-p)^{n-1} + \frac{1}{p}\sum_{n=1}^{\infty}(1-p)^{n-1}$

$= \frac{2-p}{p^2} - \frac{2}{p^2} + \frac{1}{p^2}$

$= \frac{1-p}{p^2}$

$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$

$\sum_{n=1}^{\infty} nx^{n} = \frac{x}{(1-x)^2}$

$\frac{d}{dx}\sum_{n=1}^{\infty} nx^{n} = \frac{d}{dx}\frac{x}{(1-x)^2}$

$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{1+x}{(1-x)^3}$

put $x=(1-p)$,

$\sum_{n=1}^{\infty} n^2(1-p)^{n-1} = \frac{2-p}{p^3}$

$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2}$

$\sum_{n=1}^{\infty} (1-p)^{n-1} = \frac{1}{p}$

e.g. Printing document

probability of jamming for printing one page = 0.05% = 0.0005

Expect value $= \frac{1}{p} = 2000$

We expect the printer to have one jamming on 2000-th page.