THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520C University Mathematics for Applications
Suggested Solution to Assignment 5

Ex 11.1: 27. **RELIABILITY** Suppose that a particular printer at your school has a 99.95% chance of successfully printing out a randomly selected page in a document and a 0.05% chance of jamming.

a. Find the probability that the printer will jam on the tenth page of a 10-page document after printing out the first 9 pages without jamming.

b. On what page would you expect the printer to jam when a large document is printed out?

c. Determine the probability that the printer completes a document of 25 pages without jamming.

Solution: **a.** Note that the first 9 pages have already printed successfully, whether the tenth page will be printed successfully is independent with it. Therefore the probability that the printer will jam on the tenth page is 0.05%.

b. In here, the probability that the printer jamon a page is $p = 0.05\%$, let X denote the page variable. From the lecture on geometric random variables, we know that

$$E(X) = \frac{1}{p} = \frac{1}{0.05\%}$$

Therefore the expected page with jamming is

$$E(p) = \frac{1}{0.05\%} = 2000$$

c. $p = (99.95\%)^{25} = 0.98757$

Ex 11.1: 36 **GAMES OF CHANCE** You and a friend take turns rolling a die until one of you wins by rolling a 3 or a 4. If your friend rolls first, find the probability that you will win.

Solution: Let X denote the number of game you play. Since your friend rolls first, you can't win at odd number games. Let p denote the probability of rolling a 3 or 4 and

$p = \frac{2}{6} = \frac{1}{3}$. The probability that you will win is

$$\begin{aligned} P &= P(X=2) + P(X=4) + P(X=6) + \cdots + P(X=2n) + \cdots \\ &= \frac{2}{3} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^5 \cdot \frac{1}{3} + \cdots + \left(\frac{2}{3}\right)^{2n-1} \cdot \frac{1}{3} + \cdots \\ &= \frac{2}{9}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \frac{4}{9}^n + \cdots\right) \\ &= \frac{2}{9}\frac{1}{1-4/9} = \frac{2}{5} \end{aligned}$$

Ex 11.2: 42. **USEFUL LIFE OF A MACHINE** The useful life X of a particular kind of machine is a random variable with density function

$$f(x) = \begin{cases} \frac{3}{28} + \frac{3}{x^2} & \text{if } 3 \le \text{x} \le 7 \\ 0 & \text{otherwise} \end{cases}$$

where x is the number of years a randomly selected machine stays in use.

a. Find the probability that a randomly selected machine will be useful for more than 4 years.

b. Find the probability that a randomly selected machine will be useful for less than 5 years.

c. Find the probability that a randomly selected machine will be useful between 4 and 6 years.

Solution: **a.**

$$\begin{aligned} P(X \ge 4) &= \int_4^{+\infty} f(x)\,dx = \int_4^7 \left(\frac{3}{28} + \frac{3}{x^2}\right) dx \\ &= \left(\frac{3x}{28} - \frac{3}{x}\right)\Bigg|_4^7 = \frac{9}{14} \end{aligned}$$

b.

$$\begin{aligned} P(X \le 5) &= \int_0^5 f(x)\,dx = \int_3^5 \left(\frac{3}{28} + \frac{3}{x^2}\right) dx \\ &= \left(\frac{3x}{28} - \frac{3}{x}\right)\Bigg|_3^5 = \frac{43}{70} \end{aligned}$$

c.

$$\begin{aligned} P(4 \le X \le 6) &= \int_4^6 f(x)\,dx = \int_4^6 \left(\frac{3}{28} + \frac{3}{x^2}\right) dx \\ &= \left(\frac{3x}{28} - \frac{3}{x}\right)\Bigg|_4^6 = \frac{13}{28} \end{aligned}$$

Ex 11.2: 57. **BIOLOGY** Let X be a random variable that measures the age of a randomly selected cell in a particular population. Suppose X is distributed exponentially with a probability density function of the form

$$f(x) = \begin{cases} ke^{-kx} & \text{for x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where x is the age of a randomly selected cell (in days) and k is a positive constant. Experiments indicate that it is twice as likely for a cell to be less than 3 days old as it is for it to be more than 3 days old.

a. Use this information to determine k.

b. Find the probability that a randomly selected cell is at least 5 days old.

Solution: **a.** Since experiments indicate that it is twice as likely for a cell to be less than 3 days old as it is for it to be more than 3 days old, that implies

$$\int_0^3 f(x)\,dx = 2\int_3^{+\infty} f(x)\,dx$$

That is

$$-e^{-kx}\Big|_0^3 = -2e^{-kx}\Big|_3^{+\infty} \Rightarrow 1 - e^{-3k} = 2e^{-3k} \Rightarrow k = \frac{\ln 3}{3}$$

Therefore $k \approx 0.3662$

b.

$$\begin{aligned} P(X \geq 5) &= \int_5^{+\infty} f(x)\,dx = \int_5^{+\infty} 0.3662e^{-0.3662x}\,dx \\ &= -e^{-0.3662x}\Big|_5^{+\infty} \approx 0.1602 \end{aligned}$$

Ex 11.3: 25. **MEDICAL RESEARCH** A group of patients with a potentially fatal disease has been treated with an experimental drug. Assume the survival time X for a patient receiving the drug is a random variable exponentially distributed with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if x} \geq 0 \\ 0 & \text{if x} < 0 \end{cases}$$

where x is the number of years a patient survives after first receiving the drug.

a. Research indicates that the expected survival time for a patient receiving the drug is 5 years. Based on this information, what is λ?

b. Using the value of λ determined in part (a), what is the probability that a randomly selected patient survives for less than 7 years?

c. What is the probability that a randomly selected patient survives for more than 7 years?

Solution: **a.** We know from the research that

$$\int_0^{+\infty} xf(x)\,dx = \int_0^{+\infty} \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx$$
$$= -xe^{-\lambda x}\Big|_0^{+\infty} - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^{+\infty} = \frac{1}{\lambda} = 5$$
$$\Longrightarrow \lambda = \frac{1}{5}$$

b.

$$P(X \leq 2) = \int_0^2 \frac{1}{5}e^{-\frac{1}{5}x}\,dx = -e^{-x/5}\Big|_0^2 \approx 0.32968$$

c.

$$P(X \geq 7) = \int_7^{+\infty} \frac{1}{5}e^{-\frac{1}{5}x}\,dx = -e^{-x/5}\Big|_7^{+\infty} \approx 0.2466$$

Ex 11.3: 31. Show that a uniformly distributed random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if a} \leq \text{x} \leq \text{b} \\ 0 & \text{otherwise} \end{cases}$$

has expected value

$$E(X) = \frac{a+b}{2}$$

Solution:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)\,dx = \int_a^b \frac{x}{b-a}\,dx$$
$$= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

Ex 11.3: 32. Show that the uniformly distributed random variable in Exercise 31 has variance

$$Var(X) = \frac{(b-a)^2}{12}$$

Solution:

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} [x - E(X)]^2 f(x)\, dx = \int_a^b \left(x - \frac{a+b}{2} \right)^2 \frac{1}{b-a}\, dx \\
&= \frac{1}{b-a} \int_a^b \left(x^2 - (a+b)x + \frac{(a+b)^2}{4} \right) dx \\
&= \frac{1}{b-a} \left(\frac{x^3}{3} - \frac{(a+b)x^2}{2} + \frac{(a+b)^2 x}{4} \right) \Bigg|_a^b \\
&= \frac{b^2 + ab + b^2}{3} - \frac{(a+b)^2}{2} + \frac{(a+b)^2}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$