THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2014-2015
Test 1, 12 Feb, 2015

- Time allowed: 45 minutes
- Answer all questions.
- Show your work clearly and concisely in your answer book.
- Write down your name and student ID number on the front page of your answer book.
- You are allowed to use a calculator in this test.

1. Evaluate each of the following limits.

   (a) $\lim_{n\to\infty} \frac{4n^2+3}{-n^2-5n+2}$.

   (b) $\lim_{n\to\infty} (1+\frac{1}{n+1})^{2n}$.

(12 points)

2. By using sandwich theorem, find the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt[4]{n^4+1}} + \frac{1}{\sqrt[4]{n^4+2}} + \cdots + \frac{1}{\sqrt[4]{n^4+n}}.$$

(15 points)

3. Evaluate each of the following limits.

   (a) $\lim_{x\to 0} \frac{\sin 2x \tan 3x}{x^2}$.

   (b) $\lim_{x\to +\infty} \frac{e^{x+1}+e^{-x}}{e^{x-1}-e^{-x}}$.

   (c) $\lim_{x\to -\infty} \frac{x}{\sqrt{9x^2+1}}$.

(18 points)

4. Let $f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$.

   (a) Prove that f is differentiable at $x = 0$ and find $f'(0)$.

   (b) Is $f'(x)$ differentiable at $x = 0$?

(20 points)

5. By using mean value theorem, prove that for any $x > y > 0$

$$e^y(x-y) < e^x - e^y < e^x(x-y).$$

(15 points)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

- $f(x+y) = f(x)f(y)$ for all real numbers x and y;
- $1 - x \leq f(x) \leq 1 - xf(x)$ for all real numbers x.

(a) Show that

(i) $f(0) = 1$,

(ii) $f(x) > 1$ for $x < 0$,

(iii) $f(x) > 0$ for all real number x.

Hence, deduce that $f(x)$ is strictly decreasing, that means if $a > b$, then $f(a) < f(b)$.

(b) Show that if $h > -1$, we have

$$1 - h \leq f(h) \leq \frac{1}{1+h}.$$

Hence, show that $f(x)$ is continuous at $x = 0$.

(c) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

(20 points)