MATH 4010

Tutorial 4

Eg 1. $(\ell^p)^* \cong \ell^q$, for $1<p<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$

Remark: Here and henceforth, "$\cong$" is in the sense of isomorphism of normed space (i.e. there exists a bijective linear operator

$$T: X \to Y \quad \text{s.t. } \|Tx\| = \|x\|, \ \forall x \in X.$$

Pf: Idea of proof

Step 1: $(\ell^p)^* \subset \ell^q$

Construct a injective linear operator

$$T: (\ell^p)^* \to \ell^q \quad \text{s.t.} \quad \|Tf\|_{\ell^q} \leq \|f\|$$

Step 2: $\ell^q \subset (\ell^p)^*$

To show T is surjective and verify $\|Tf\|_{\ell^q} = \|f\|$

Now, we give the proof.

(i) Let $f \in (\ell^p)^*$. [We need to find an unique y_f (determined by f) such that $\|y_f\|_{\ell^q} \leq \|f\|$]

Since $e_k = \{\delta_{kj}\}$, which is a sequence with k-th term 1, others zero, is a Schauder basis of ℓ^p, there exist an unique sequence of real number ξ_k s.t. $x = \sum_{k=1}^{\infty} \xi_k e_k$, $\forall x \in \ell^p$.

Then, $f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k)$, since f is continuous and linear.

Set $\eta_k = f(e_k)$. Then $f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k$

It sufficies to show $y_f = \{\eta_k\} \in \ell^q$ and $\|y_f\|_{\ell^q} \leq \|f\|$.

Indeed, $\forall n \in \mathbb{N}$, we can construct a sequence $x_n = \{\xi_k^{(n)}\}$ as

$$\xi_k^{(n)} = \begin{cases} |\eta_k|^q / \eta_k & \text{if } k \leq n \text{ and } \eta_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \eta_k = 0 \end{cases}$$

Then, it is clear that $x_n \in \ell^p$, since it has only finite nonzero term and $f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \eta_k = \sum_{k=1}^{n} |\eta_k|^q$

By the boundness of f, one has

$$\sum_{k=1}^{n} |\eta_k|^q = |f(x_n)| \leq \|f\| \|x_n\|_{\ell^p} = \|f\| \left(\sum_{k=1}^{\infty} |\xi_k^{(n)}|^p\right)^{\frac{1}{p}}$$

$$\leq \|f\| \left(\sum_{k=1}^{n} |\eta_k|^{(q-1)p}\right)^{\frac{1}{p}} = \|f\| \left(\sum_{k=1}^{n} |\eta_k|^q\right)^{\frac{1}{p}}, \text{ since } \frac{1}{p}+\frac{1}{q}=$$

So, $\left(\sum_{k=1}^{n}|\eta_k|^q\right)^{\frac{1}{q}} \leq \|f\|$.

Letting $n\to\infty$, one has $\|y_f\|_{\ell^q} = \left(\sum_{k=1}^{\infty}|\eta_k|^q\right)^{\frac{1}{q}} \leq \|f\|$.

Therefore, we have construct an injective linear operator (it is easy to check the linearity!)

$$T: (\ell^p)^* \to \ell^q \quad \text{by} \quad f(x) = \sum_{k=1}^{\infty}\xi_k\eta_k, \ \forall x = \{\xi_k\}$$
$$f \mapsto y. \qquad \text{with } y = \{\eta_k\} = \{f(e_k)\}.$$

Moreover, $\|y\|_{\ell^q} \leq \|f\|$.

(ii) $\forall\, z \in \{\zeta_k\} \in \ell^q$, define a mapping g as follows

$$g(x) = \sum_{k=1}^{\infty}\xi_k\zeta_k, \ \forall x = \{\xi_k\} \in \ell^p.$$

Then, it is obvious that g is linear.

Moreover, the Hölder inequality yields that

$$|g(x)| \leq \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\infty}|\zeta_k|^q\right)^{\frac{1}{q}} = \|x\|_{\ell^p}\|z\|_{\ell^q},$$

Which implies that $\|g\| \leq \|z\|_{\ell^q} < +\infty$, so g is bounded.

Therefore, T is also a surjective.

And, by Hölder inequality, $\|f\| \leq \|y\|_{\ell^q}$.

Therefore, T is an isomorphism, i.e. $(\ell^p)^* \cong \ell^q$, for $1<p<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$.

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Eg 2. Let c_0 be the space of all sequences which converges to zero. The norm on c_0 is given by $\|\{\xi_k\}\|_{c_0} = \sup_k|\xi_k|$. Show that $(c_0)^* \cong \ell^1$

Pf: (i) $\forall f \in (c_0)^*$, $f(x) = \sum_{k=1}^{\infty}\xi_k f(e_k)$ with $x = \sum_{k=1}^{\infty}\xi_k e_k \in c_0$ and $\xi_k \to 0$ as $k\to\infty$.

Set $\eta_k = f(e_k)$, then $f(x) = \sum_{k=1}^{\infty}\xi_k\eta_k$.

It sufficies to show $y = \{\eta_k\} \in \ell^1$ and $\|y\|_{\ell^1} \leq \|f\|$.

Indeed, we can construct a sequence $x_n = \{\xi_k^{(n)}\}$ as

$$\xi_k^{(n)} = \begin{cases} |\eta_k|/\eta_k & \text{if } k<n \text{ and } \eta_k \neq 0 \\ 0 & \text{if } k \geq n \text{ or } \eta_k = 0 \end{cases}$$

Then, it is clear that $x_n \in c_0$, and

$$f(x_n) = \sum_{k=1}^{\infty}\xi_k^{(n)}\eta_k = \sum_{k=1}^{n}|\eta_k|$$

and $|f(x_n)| \leq \|f\|\|x_n\|_{c_0} = \|f\|$, since f is bound

So, $\sum_{k=1}^{n}|\eta_k| \leq \|f\|$. Letting $n\to\infty$, one has $y \in \ell^1$ and $\|y\|_{\ell^1} \leq \|f\|$

(ii) $\forall\ z = \{\zeta_k\} \in \ell^1$, define $g: c_0 \to \mathbb{R}$ as

$$g(x) = \sum_{k=1}^{\infty} \xi_k \zeta_k \ , \ \forall\ x = \{\xi_k\} \in c_0$$

Then $|g(x)| \leqslant \sup_k |\xi_k| \sum_{k=1}^{\infty} |\eta_k| \leq \|x\|_{c_0} \|y\|_{\ell^1}$

So, $\|g\| \leqslant \|y\|_{\ell^1} < +\infty$, i.e. g is bounded.

It is obvious g is linear.

Therefore $T: (c_0)^* \to \ell^1$ defined by $y = \{\eta_k\} := \{f(e_k)\}$ s.t.

$f \mapsto y$ $\qquad\qquad f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k$

is a bijective linear operator and $\|y\|_{\ell^1} = \|f\|$.

That is, $(c_0)^* \cong \ell^1$.

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