

• Examples of Banach spaces

Ex 1. Euclidean space \mathbb{R}^n with standard norm

$$x = (x_1, x_2, \dots, x_n) \mapsto \|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$$

$(\mathbb{R}^n, \|\cdot\|)$ is a Banach space.

Ex 2. $L^p(\Omega, \mathcal{F}, \mu)$ is a Banach space, where $1 \leq p < \infty$.

• For $1 \leq p < \infty$, $L^p(\Omega, \mathcal{F}, \mu) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ is measurable and } \int_{\Omega} |f(x)|^p d\mu < +\infty\}$

$$\text{with } \|f\|_{L^p} (= \|f\|_p) = \left(\int_{\Omega} |f(x)|^p d\mu\right)^{\frac{1}{p}}$$

• For $p = \infty$, $L^\infty(\Omega, \mathcal{F}, \mu) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ is measurable and } |f(x)| \leq M \text{ a.e. on } \Omega \text{ for some constant } 0 < M < \infty\}$

$$\text{with } \|f\|_{L^\infty} (= \|f\|_\infty) = \inf \{M : |f(x)| \leq M \text{ a.e. on } \Omega\}$$

Here $(\Omega, \mathcal{F}, \mu)$ denotes a measure space, i.e. Ω is a set and

(i) \mathcal{F} is a σ -algebra in Ω , i.e. \mathcal{F} is a collection of subsets of Ω s.t.

(a) $\emptyset \in \mathcal{F}$;

(b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(c) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ whenever $A_n \in \mathcal{F}, \forall n$.

(ii) μ is a measure, i.e. $\mu: \mathcal{F} \rightarrow [0, \infty]$ satisfies

(a) $\mu(\emptyset) = 0$

(b) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\}$ is a disjoint countable subsets of \mathcal{F} .

(iii) Ω is σ -finite, i.e. \exists countable subsets $\{\Omega_n\}$ of \mathcal{F} s.t.

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n \text{ and } \mu(\Omega_n) < +\infty, \forall n.$$

Remarks: 1° Recall that for $f, g \in L^1(\Omega, \mu)$, $f \sim g$ if $f = g$ a.e. on Ω .

2° For $f \in L^\infty(\Omega, \mu)$, $|f(x)| \leq \|f\|_\infty$ a.e. on Ω .

Indeed, by the def. of inf., $\forall n \in \mathbb{N}$,

$$|f(x)| \leq \|f\|_\infty + \frac{1}{n} \text{ a.e.} \Leftrightarrow \mu(E_n) = 0 \text{ with } E_n = \{x \in \Omega : |f(x)| > \|f\|_\infty + \frac{1}{n}\}$$

Note that $E := \{x \in \Omega : |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} E_n$

Then, $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$.

3°, $\Omega = \mathbb{R}^n$, μ - Lebesgue measure $\Rightarrow L^1(\mathbb{R}^n)$

$\Omega = \mathbb{Z}^+$, μ - counting measure $\Rightarrow l^1$.

Pf: Step 1: $L^p(\Omega, \mathcal{F}, \mu)$ is a normed space.

It is clear that (i) $\|f\|_{L^p} \geq 0$ and $\|f\|_{L^p} = 0$ iff $f = 0$ a.e. (Note that Equivalent class is introduced as shown in Remark 1°)

$$(ii) \|\alpha f\|_{L^p} = |\alpha| \|f\|_{L^p}$$

It suffices to show the triangle inequality holds true, i.e.

$$(M) \quad \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

This is the well-known Minkowski inequality.

This is obvious for $p=1$ or ∞ . W.L.O.G we assume $1 < p < \infty$.

To prove it, we first prove the Hölder inequality, i.e.

$$(H) \quad \|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 < p, q < \infty \text{ (} 1 \leq p, q \leq \infty \text{)}$$

(H) is a consequence of Young inequality

$$(Y) \quad ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \text{ for } a, b \geq 0.$$

Indeed, by applying Young inequality,

$$\|fg\|_{L^1} = \int_{\Omega} |f||g| d\mu \leq \int_{\Omega} \left[\frac{1}{p} |f|^p + \frac{1}{q} |g|^q \right] d\mu = \frac{1}{p} \|f\|_{L^p}^p + \frac{1}{q} \|g\|_{L^q}^q.$$

Thus, if $\|f\|_{L^p} = \|g\|_{L^q} = 1$, then $\|fg\|_{L^1} \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_{L^p} \|g\|_{L^q}$

Otherwise, replacing f by $\frac{f}{\|f\|_{L^p}}$ and g by $\frac{g}{\|g\|_{L^q}}$

$$\text{we have, } \left\| \frac{f}{\|f\|_{L^p}} \right\|_{L^p} = 1, \left\| \frac{g}{\|g\|_{L^q}} \right\|_{L^q} = 1 \text{ and } \left\| \frac{f}{\|f\|_{L^p}} \cdot \frac{g}{\|g\|_{L^q}} \right\|_{L^1} \leq \frac{1}{p} + \frac{1}{q} = 1$$

The Young inequality follows from the concavity of \log fn, i.e.

$$\log\left(\frac{1}{p} a^p + \frac{1}{q} b^q\right) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q \geq \log a + \log b = \log ab.$$

Now, we prove (M),

$$\forall f, g \in L^p,$$

$$\begin{aligned} \|f+g\|_{L^p}^p &= \int_{\Omega} |f+g|^p d\mu \leq \int_{\Omega} |f+g|^{p-1} |f| d\mu + \int_{\Omega} |f+g|^{p-1} |g| d\mu \\ &\leq \left[\int_{\Omega} (|f+g|^{p-1})^q \right]^{\frac{1}{q}} \|f\|_{L^p} + \left[\int_{\Omega} (|f+g|^{p-1})^q \right]^{\frac{1}{q}} \|g\|_{L^p} \\ &= \|f+g\|_{L^p}^{p-1} \|f\|_{L^p} + \|f+g\|_{L^p}^{p-1} \|g\|_{L^p} \end{aligned}$$

$$\text{Since } \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$$

$$\text{So, } \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Step 2. $L^p(\Omega, \mathcal{F}, \mu)$ is complete.

Note that the induced metric, $d(f, g) := \|f - g\|_{L^p}$.

Case 1: $1 \leq p < \infty$.

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\Omega, \mathcal{F}, \mu)$.

It suffices to show that there exist a subsequence $\{f_{n_k}\}$ converges in L^p .

We extract a subsequence $\{f_{n_k}\}$ satisfying

$$\|f_{n_{k+1}} - f_{n_k}\|_{L^p} \leq \frac{1}{2^k}, \quad \forall k \geq 1.$$

Now, we prove f_{n_k} converges in L^p .

$$\text{Define } f(x) = f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$$

$$g(x) = |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

Then, the partial sum. of f, g are

$$S_K f = f_{n_1} + \sum_{k=1}^{K-1} (f_{n_{k+1}} - f_{n_k}) = f_{n_K}$$

$$S_K g = |f_{n_1}(x)| + \sum_{k=1}^{K-1} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

$$\text{Note that } \|S_K g\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K-1} \|f_{n_{k+1}} - f_{n_k}\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{K-1} 2^{-k}$$

By the monotone convergence theorem (M.C.T.)

$$S_K g \rightarrow g \text{ a.e. on } \Omega \text{ and } \int_X g^p d\mu < +\infty.$$

So, f converges a.e. and $f \in L^p$ because of $|f(x)| \leq g(x)$ a.e.

To prove $f_{n_k} \rightarrow f$ in L^p , as $k \rightarrow \infty$

$$\|f_{n_k} - f\|^p = \|S_K f - f\|^p \leq 2^p (\|S_K g\|^p + \|f\|^p) \\ \leq 2^{p+1} \|g\|^p$$

and $f_{n_k} \rightarrow f$ a.e. as $k \rightarrow \infty$

By D.C.T., $\|f_{n_k} - f\|_{L^p} \rightarrow 0$ as $k \rightarrow \infty$.

Case 2. $p = \infty$

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^\infty(\Omega, \mathcal{F}, \mu)$.

Then, $\forall k \in \mathbb{N}, \exists N_k$ s.t. $\forall m, n \geq N_k, \|f_n - f_m\|_{L^\infty} < \frac{1}{k}$.

i.e. $|f_n(x) - f_m(x)| < \frac{1}{k}$ on $\Omega \setminus E_k$ with $\mu(E_k) = 0$.
 $\forall n, m \geq N_k$

Step 2. $L^p(\Omega, \mathcal{F}, \mu)$ is complete.

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and $f_{n_k} \rightarrow f$ a.e. as $K \rightarrow \infty$

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