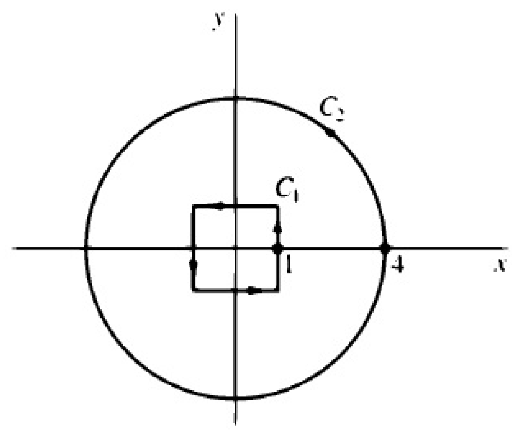
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Spring MATH2230
Homework Set 7 (Due on Mar. 12)

All the homework problems are taken from Complex Variables and Applications, Ninth Edition, by James Ward Brown/Ruel V. Churchill.

P159-161

2 . Let C_1 demote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. With the aid of the corollary in Sec. 53. point out why

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



when

(a) $f(z) = \dfrac{1}{3z^2+1}$ (b) $f(z) = \dfrac{z+2}{\sin(z/2)}$

(c) $f(z) = \dfrac{z}{1-e^z}$.

3 . If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z-z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2 ... \\ 2\pi i & \text{when } n = 0. \end{cases}$$

according to Exercise 13. Sec. 46.. Use that result and the corollary in Sec. 53 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$ described in the positive sense, then

$$\int_{C} (z-2-i)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2 ... \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4 . Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in figure can be written

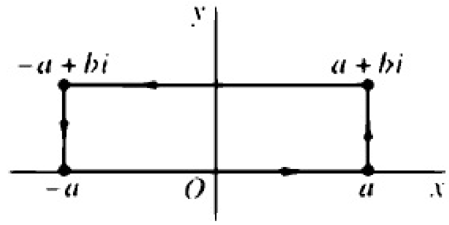
$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy.$$

Thus, with the aid of the Cauchy-Goursat theorem. show that

$$\int_0^a e^{-x^2} \cos 2bx dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay dy$$



(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left| \int_0^b e^{y^2} \sin 2ay dy \right| \le \int_0^b e^{y^2} dy$$

obtain the desired integration formula by letting ll tend to infinity in the equation at the end of part (a).

5 . According to Exercise 6. Sec. 43. the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

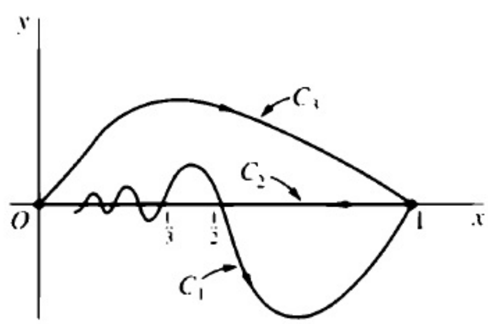
$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2. Apply the Cauchy-Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)dz = \int_{C_3} f(z)dz \quad \text{and} \quad \int_{C_2} f(z)dz = -\int_{C_3} f(z)dz.$$

Conclude that even though the dosed contour $C = C_1 + C_2$intersects itself an infinite number of times,

$$\int_C f(z)dz = 0$$



6 . Let C denote the positively oriented boundary of the half disk $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, -\pi/2 < \theta < 3\pi/2)$$

of the multiple-valued function $z^{1/2}$. Show that

$$\int_C f(z)dz = 0$$

P170

1 . Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}dz}{z - (\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z dz}{z(z^2+8)}$; (c) $\displaystyle\int_C \frac{zdz}{2z+1}$; (d) $\displaystyle\int_C \frac{\cosh z dz}{z^4}$;

(e) $\displaystyle\int_C \frac{\tan(z/2)dz}{(z-x_0)^2} \quad (-2 < x_0 < 2)$.

2 . Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

$$(a) \quad g(z) = \frac{1}{z^2+4}; \quad (b) \quad g(z) = \frac{1}{(z^2+4)^2}.$$

3 . Let C be the circle $|z| = 3$ described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds \quad (|z| \neq 3)$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4 . Let C be any simple closed contour described in the positive sense in the z plane and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3} ds$$

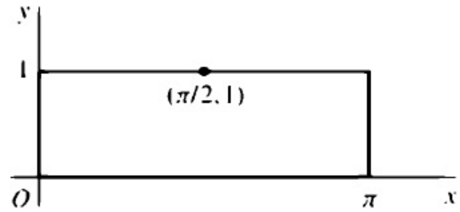
Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

P172

10 . Let f be an entire function such that $|f(z)| \leq A|z|$ for all z where A is a fixed positive number. Show that $f(z) = a_1 z$ where a_1 is a complex constant.
Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

P177-178

4 . Let R region $0 \leq x \leq \pi$, $0 \leq y \leq 1$. Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = (\pi/2) + i$.
Suggestion : Write $|f(z)|^2 = \sin^2 x + \sinh^2 y$ (see Sec. 37) and locate points in R at which $\sin^2 x$ and $\sinh^2 y$ are the largest.



6 . Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Illustrate results in Sec. 59 and Exercise 5 by finding points in R where the component function $u(x, y) = Re|f(z)|$ reaches its maximum and minimum value.

(Sec. 59 and Exercise 5 (no need to do, just for reference): Let $f = u + iv$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function u has a minimum value in R which occurs on the boundary of R and never in the interior.)