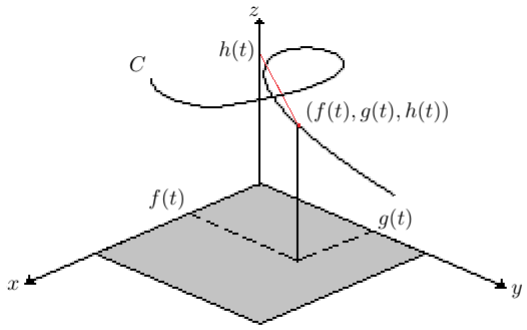
MATH2010E LECTURE 4: CURVES

1. Parametrization of a curve

A continuous curve in $\mathbb{R}^3$ can be parametrized by

$$C = \{(f(t), g(t), h(t)) | t \in \mathbb{R}\}. \tag{1.1}$$

Here $f(t)$, $g(t)$ and $h(t)$ are continuous functions. One can regard t as the time variable. Then this parametrization gives us the trajectory of the position of a particle in $\mathbb{R}^3$.



Pic. 1

Similar to the parametrization of a line in $\mathbb{R}^3$, the choice of parameter is not unique for curves in $\mathbb{R}^3$.

In the case that f, g and h are differentiable, then we call C a differentiable curve in $\mathbb{R}^3$; If their derivatives exist for any order, we call C a smooth curve. In this lecture, we only consider differentiable or smooth curves.

Now if we regard $p(t) = (f(t), g(t), h(t))$ as the position function for a particle. Then we can define the **velocity** (or tangent vector) at time s for this particle to be

$$\vec{v}(s) = p'(s) = \Big(\frac{df}{dt}(s), \frac{dg}{dt}(s), \frac{dh}{dt}(s)\Big) \tag{1.2}$$

and define the **speed** for this particle to be $|\vec{v}(s)|$.

By fundamental theorem of calculus, we have

$$p(s) = \int_0^s \vec{v}(t)dt + p(0) \tag{1.3}$$

where the integration is defined by

$$\int_a^b \vec{v}(t)dt := \left(\int_a^b v_1(t)dt, \int_a^b v_2(t)dt, \int_a^b v_2(t)dt \right) \tag{1.4}$$

when $\vec{v}(t) = (v_1(t), v_2(t), v_3(t)$.

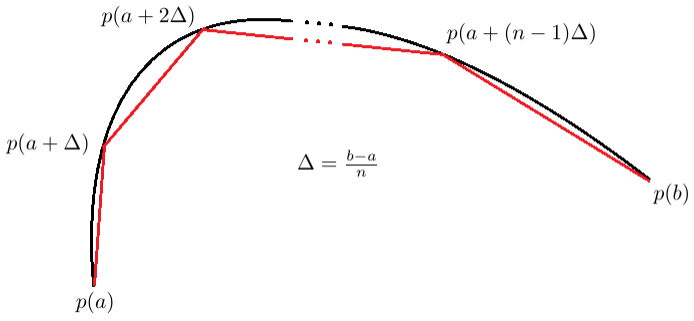
We can see that, both p and v are **vector value** functions, i.e. functions map from $\mathbb{R}$ to $\mathbb{R}^3$. These functions have the following basic properties.

Proposition 1.1. Let $\vec{v}(t), \vec{u}(t)$ be two vector value functions and $r(t)$ be a real value function. Then
1. $\frac{d}{dt}(r\vec{v}) = \frac{dr}{dt}\vec{v} + r\frac{d\vec{v}}{dt}$;
2. (Linearity) $\frac{d}{dt}(\vec{v} + c\vec{u}) = \frac{d\vec{v}}{dt} + c\frac{d\vec{u}}{dt}$;
3. (Dot product) $\frac{d}{dt}(\vec{v} \cdot \vec{u}) = (\frac{d\vec{v}}{dt} \cdot \vec{u}) + (\vec{v} \cdot \frac{d\vec{u}}{dt})$;
4. (Cross product) $\frac{d}{dt}(\vec{v} \times \vec{u}) = (\frac{d\vec{v}}{dt} \times \vec{u}) + (\vec{v} \times \frac{d\vec{u}}{dt})$;
5. (Chain rule) $\frac{d}{dt}(\vec{v} \circ r) = \frac{dr}{dt}(\frac{d\vec{v}}{dt} \circ r)$.

Arc length of space curves. To compute the arc length for a curve, we should integrate $|\vec{v}|$ instead. If we want to know the arc length of C from $p(a)$ to $p(b)$, $a < b$, then we should compute

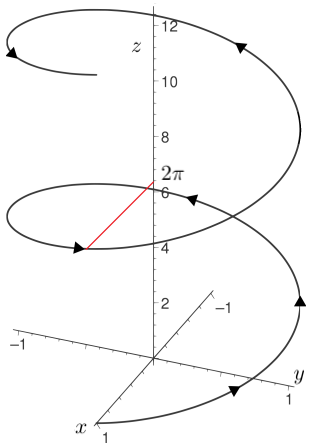
$$l_{ab} := \int_a^b |\vec{v}|(t)dt. \tag{1.5}$$

One can regard this integral as the limit of Riemann sum of the following graph.



Pic. 3

Example.(Helix) Let $p(t) = (\cos(t), \sin(t), t)$, then the arc length for this curve form $t = 0$ to $t = 2\pi$ will be



Pic. 3
(Wrok by RobHar: https://fi.wikipedia.org/wiki/Kierre♯/media/File:Helix.svg)

$$l_{0,2\pi} = \int_0^{2\pi} |\vec{v}|(t)dt = \int_0^{2\pi} \sqrt{(-\sin)^2(t) + \cos^2(t) + 1^2}dt \tag{1.6}$$
$$= \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi.$$

Remark 1.2. Recall that the parametrization of a curve is not unique. In particular, we can always find the parameter such that $|\vec{v}| \equiv 1$. In this case, the arc length

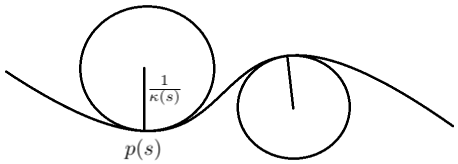
$$l_{ab} = \int_a^b 1dt = b - a.$$

2. Plane curves: Curvature and normal vector

Here we consider a smooth plane curve $p : \mathbb{R} \to \mathbb{R}^2$, $p(t) = (X(t), Y(t))$. Suppose the parameter we choose satisfies $|\vec{v}| = |p'| \equiv 1$. Then we define

$$\kappa(s) := |p''|(s) = \left|\left(\frac{d^2X}{dt^2}(s), \frac{d^2Y}{dt^2}(s)\right)\right| \tag{2.1}$$
$$= \sqrt{\left(\frac{d^2X}{dt^2}(s)\right)^2 + \left(\frac{d^2Y}{dt^2}(s)\right)^2}.$$

Then the function κ have the following geometric meaning: For any point $p(s)$ on the curve, there exists a unique circle approximates the curve at $p(s)$ up to the second order derivatives. The radius of this circle will be $\frac{1}{\kappa(s)}$. We call κ the **curvature** of this curve.



Pic. 4

We call this circle **osculating circle** of this curve.

To show this fact, let $\mathcal{C}(t)$ be the parametrization of the osculating circle at point $p(s)$. So by definition, we can assume

$$\mathcal{C}(s) = p(s); \tag{2.2}$$

$$|\mathcal{C}'|(t) = |p'|(t) \equiv 1; \tag{2.3}$$

$$|\mathcal{C}''|(s) = |p''|(s) = \kappa(s) \tag{2.4}$$

by choosing a suitable parametrization. Meanwhile, because $\mathcal{C}$ is a circle, so this parametrization must be of the form

$$\mathcal{C}(t) = (R\cos(\alpha t + \beta) + p_1, R\sin((\alpha t + \beta)) + p_2). \tag{2.5}$$

We should notice that R is the radius of the circle $\mathcal{C}$.

Since $|\mathcal{C}'|(t) \equiv 1$, we have

$$1 = |(-R\alpha\sin(\alpha t + \beta), R\alpha\cos(\alpha t + \beta)| = R|\alpha|.$$

So $R = \frac{1}{|\alpha|}$. By the condition $|\mathcal{C}''|(s) = \kappa(s)$, we have

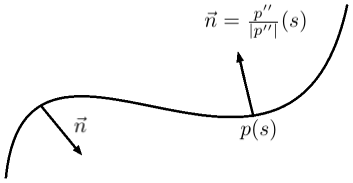
$$\kappa(s) = |(-R\alpha^2\cos(\alpha t + \beta), -R\alpha^2\sin(\alpha t + \beta))| = R|\alpha|^2 = |\alpha|.$$

Therefore, $\kappa(s) = \frac{1}{R}$ as our wish.

To find the center for a osculating circle, one should notice that the center will be on the line $L = \{p(s) + s\vec{n} | s \in \mathbb{R}\}$ for a normal vector $\vec{n}$ perpendicular to $v(s)$. This $\vec{n}$ can be found as follows.

$$0 = \frac{d}{dt}|\vec{v}|^2 = 2\frac{d\vec{v}}{dt} \cdot \vec{v}. \tag{2.6}$$

So $\frac{d\vec{v}}{dt}(s) = p''(s) \perp \vec{v}(s)$ for all $s \in \mathbb{R}$. Therefore, we can take $\vec{n} = \frac{p''}{|p''|}(s)$. This normal vector is called the **principal normal vector**. We should notice that it point to the side the curve bend.



Pic. 5

According to this property of principal normal vector, we have the center of the osculating circle at point $p(s)$ will be $p(s)+R\vec{n} = p(s)+\frac{1}{|p''|(s)}\frac{p''(s)}{|p''|(s)} = p(s)+\frac{p''(s)}{|p''|^2(s)}$. So the osculating circle can be parametrized as follows.

$$\mathcal{C} = \Big\{p(s) + \frac{p''(s)}{|p''|^2(s)} + \Big(\frac{1}{|p''|(s)}\cos(t), \frac{1}{|p''|(s)}\sin(t)\Big)\Big\}. \tag{2.7}$$

3. Space curves

In the case that $p : \mathbb{R} \to \mathbb{R}^3$ is a smooth space curve, we will have an extra direction to discuss. Suppose the parameter we choose satisfies $|p'| \equiv 1$ again. Let $p'(s) := T(s)$ be the tangent vector of the curve and $\frac{p''}{|p''|}(s) := N(s)$ be the principal normal vector. We define

$$B := T \times N \tag{3.1}$$

to be the **binormal vector** of the curve. $\{T, N, B\}$ therefore forms an orthonormal frame along the curve.

Roughly speaking, the tangent T determines the first order derivative for the curve and N determines the second derivative. All the higher order deviation will then move along the binormal direction B. One example to show this is to compute the acceleration.

Acceleration. Let $p : \mathbb{R} \to \mathbb{R}^3$ a space curve. This time we don't assume $|\frac{dp}{dt}| = 1$. Consider the curve as the trajectory of a particle, then the acceleration will be the derivative of the velocity $\vec{v} = \frac{dp}{dt}$. We denote it by $\vec{a}$.

Suppose we have $t = t(s)$ with s as the new parameter for the curve satisfying $|\frac{dp}{ds}| = 1$. Then we have

$$\begin{aligned} \vec{a} = \frac{d^2p}{dt^2} &= \frac{d}{dt}\Big(\frac{dp}{ds}\frac{ds}{dt}\Big) = \frac{d}{dt}\Big(\frac{ds}{dt}T\Big) \\ &= \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{dt} \\ &= \frac{d^2s}{dt^2}T + \Big(\frac{ds}{dt}\Big)^2\frac{dT}{ds} \\ &= \frac{d^2s}{dt^2}T + \Big(\frac{ds}{dt}\Big)^2\kappa N. \end{aligned} \tag{3.2}$$

By taking $a_T = \frac{d^2s}{dt^2}$ and $a_N = \Big(\frac{ds}{dt}\Big)^2\kappa$, we can see that $\vec{a} = a_T T + a_N N$. So it doesn't depend on B.