2017-18 MATH1010J
Lecture 21: Integration by parts
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Remark: the note is for reference only. It may contain typos. Read at your own risk.

1 Integration by Parts

Let $u(x)$ and $v(x)$ be differentiable functions. By the product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x,

$$\begin{aligned}\int u\frac{dv}{dx}\,dx &= \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx\end{aligned}$$

which is

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or

$$\int u\,dv = uv - \int v\,du$$

The process is called **Integration by parts**

For the definite integral, we also have

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx.$$

or

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

2 Examples

Example 2.1. Integrate involving e^x
Compute $\int xe^x \, dx$. ■

Answer.

$$\begin{aligned} \int xe^x \, dx &= \int x \, de^x \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C \end{aligned}$$

Example 2.2. Integration involving $\ln x$
Using integration by parts to compute

$$\int \ln x \, dx, \qquad \text{for } x > 0.$$

■

Answer. Let $u = \ln x$, $v = x$, then

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \, d \ln x \\ &= x \ln x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Example 2.3. Integration involving $\ln x$
Compute

$$\int x^2 \ln x dx$$

■

Answer.

$$\begin{aligned}
\textstyle\int x^2 \ln x \, dx &= \int (\ln x) x^2 \, dx \\
&= \int (\ln x) \frac{d}{dx} \left(\frac{x^3}{3} \right) dx \qquad (u = \ln x, v = \frac{x^3}{3}) \\
&= \int \ln x \, d\frac{x^3}{3} \\
&= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \, d(\ln x) \\
&= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} \, dx \\
&= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx \\
&= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C
\end{aligned}$$

Example 2.4. Integration involving trigonometric functions
Compute

$$\int x \cos x dx$$

■

Answer. Let $u = x$, $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$.

$$\begin{aligned}
\int x \cos x dx &= \int x d \sin x \\
&= x \sin x - \int \sin x dx \\
&= x \sin x + \cos x + C
\end{aligned}$$

Example 2.5. Integration involving Inverse trigonometric function
Evaluate

$$\int \tan^{-1} x dx.$$

■

Answer.

$$\begin{aligned}\int \tan^{-1} x dx &= x\tan^{-1} x - \int x\, d\tan^{-1} x\\ &= x\tan^{-1} x - \int \frac{x\,dx}{1+x^2}\end{aligned}$$

Let $u = 1 + x^2$, $du = 2xdu$.

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{\ln|u|}{2} + C.$$

Hence

$$\int \tan^{-1} x dx = x\tan^{-1} x - \frac{\ln(1+x^2)}{2} + C.$$

Example 2.6. Definite integral

Compute

$$\int_1^e x\ln x dx.$$

■

Answer.

$$\begin{aligned}\int_1^e x\ln x dx &= \int_1^e \ln x d\frac{x^2}{2}\\ &= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2} d\ln x\\ &= \left(\frac{e^2}{2}\ln e - \frac{1}{2}\ln 1\right) - \int_1^e \frac{x}{2}dx\\ &= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e\\ &= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)\\ &= \frac{e^2}{4} + \frac{1}{4}.\end{aligned}$$

Example 2.7. Apply integration by parts twice

Evaluate $\int x^2 e^x\, dx$. ■

Answer.

$$\begin{aligned}\int x^2 e^x\,dx &= \int x^2\,de^x \\ &= x^2e^x - \int e^x\,dx^2 \\ &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - \int 2x\,de^x \\ &= x^2e^x - 2(xe^x - \int e^x\,dx) \\ &= x^2e^x - 2(xe^x - e^x + C) \\ &= x^2e^x - 2xe^x + 2e^x + C'\end{aligned}$$

Example 2.8. Apply integration by parts twice
Evaluate $\int e^x \cos x\,dx$. ■

Answer.

$$\begin{aligned}\int e^x \cos x\,dx = \int \cos x\,de^x \\ &= e^x\cos x + \int e^x \sin x\,dx \\ &= e^x\cos x + \int \sin x\,de^x \\ &= e^x\cos x + e^x \sin x - \int e^x \cos x dx.\end{aligned}$$

We are back to the original integral, but with a **negative sign**! So

$$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x$$

$$\int e^x\cos x\,dx = \frac{1}{2}(e^x\cos x + e^x \sin x) + C.$$

Example 2.9. Apply integration by parts several times
Evaluate $\int x^3 \cos x\,dx$. ■

Answer.

$$\begin{aligned}
\int x^3 \cos x \, dx &= \int x^3 \, d \sin x \\
&= x^3 \sin x - \int \sin x \, d(x^3) = x^3 \sin x - 3 \int x^2 \sin x \, dx \\
&= x^3 \sin x + 3 \int x^2 \, d \cos x \\
&= x^3 \sin x + 3(x^2 \cos x - \int \cos x \, d(x^2)) \\
&= x^3 \sin x + 3x^2 \cos x - 6 \int x \cos x \, dx \\
&= x^3 \sin x + 3x^2 \cos x - 6 \int x \, d \sin x \\
&= x^3 \sin x + 3x^2 \cos x - 6(x \sin x - \int \sin x dx) \\
&= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C
\end{aligned}$$

3 Reduction formula

Example 3.1. *Evaluate*

$$\int x^n e^x dx$$

and compute the integral for $n = 3$. ■

Answer. Let

$$u = x^n \text{ and } dv = e^x dx.$$

Then

$$du = nx^{n-1} dx \text{ and } v = e^x.$$

By integration by parts

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

This is call the **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When $n = 0$,

$$\int e^x\,dx = e^x + C.$$

When $n = 1$,

$$\int xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

When $n = 2$,

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2\int xe^x dx \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - 3\int x^2 e^x dx \\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C\end{aligned}$$

Example 3.2. *Find a reduction formula for*

$$\int \cos^n x dx$$

and compute

$$\int_0^{\pi/2} \cos^4 x dx.$$

■

Answer. Let

$$u = \cos^{n-1} x \text{ and } dv = \cos x dx.$$

Then

$$du = (n-1)\cos^{n-2} x(-\sin x)dx \text{ and } v = \sin x.$$

Using integration by parts

$$\begin{aligned}\int \cos^n x dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x - (n-1)\int \cos^n x dx.\end{aligned}$$

Thus

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx.$$

So the final formula is

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Also

$$\begin{aligned}\int_0^{\pi/2} \cos^n x dx &= \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx.\\ &= \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx\end{aligned}$$

Hence

$$\int_0^{\pi/2} \cos^4 x dx = \frac{4-1}{4}\int_0^{\pi/2} \cos^2 x dx = \frac{4-1}{4} \times \frac{2-1}{2}\int_0^{\pi/2} dx = \frac{3\pi}{16}.$$