THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5520
Differential Equations & Linear Algebra
Suggested Solution for Assignment 1
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Exercise 1.1 Question 1(a)

$$y' + y = 4e^{3x}$$

Solution: An integrating factor is

$$\exp\left(\int 1\,dx\right) = e^x.$$

Multiplying both sides by e^x, we have

$$\begin{aligned}
e^x y' + e^x y &= 4e^{4x} \\
\frac{d}{dx}(e^x y) &= 4e^{4x} \\
e^x y &= \int 4e^{4x}\,dx \\
e^x y &= e^{4x} + C \\
y &= e^{3x} + Ce^{-x}.
\end{aligned}$$

□

Exercise 1.1 Question 2(c)

$$(x^2+4)y' + 3xy = 3x; \quad y(0) = 3$$

Solution: Dividing both sides by $x^2 + 4$, the equation becomes

$$y' + \frac{3x}{x^2+4}y = \frac{3x}{x^2+4}.$$

An integrating factor is

$$\exp\left(\int \frac{3x}{x^2+4}\,dx\right) = \exp\left(\frac{3}{2}\ln(x^2+4)\right) = (x^2+4)^{\frac{3}{2}}.$$

Multiplying both sides by $(x^2+4)^{\frac{3}{2}}$, we have

$$\begin{aligned}
(x^2+4)^{\frac{3}{2}}y' + 3x(x^2+4)^{\frac{1}{2}}y &= 3x(x^2+4)^{\frac{1}{2}} \\
\frac{d}{dx}((x^2+4)^{\frac{3}{2}}y) &= 3x(x^2+4)^{\frac{1}{2}} \\
(x^2+4)^{\frac{3}{2}}y &= \int 3x(x^2+4)^{\frac{1}{2}}\,dx \\
(x^2+4)^{\frac{3}{2}}y &= (x^2+4)^{\frac{3}{2}} + C \\
y &= 1 + C(x^2+4)^{-\frac{3}{2}}.
\end{aligned}$$

Since $y(0) = 3$, we have

$$3 = 1 + \frac{1}{8}C \implies C = 16.$$

The solution of the initial value problem is

$$y = 1 + 16(x^2 + 4)^{-\frac{3}{2}}.$$

□

Exercise 1.2 Question 2(a)

$$xy' - y = 2x^2y; \quad y(1) = 1$$

Solution:

$$\begin{aligned} xy' - y &= 2x^2y \\ xy' &= (2x^2 + 1)y \\ \frac{y'}{y} &= 2x + \frac{1}{x} \\ \frac{dy}{y} &= \left(2x + \frac{1}{x}\right) dx \\ \int \frac{dy}{y} &= \int \left(2x + \frac{1}{x}\right) dx \\ \ln|y| &= x^2 + \ln x + C' \\ y &= Cxe^{x^2}, \text{ where } C = \pm e^{C'}. \end{aligned}$$

Since $y(1) = 1$, we have

$$1 = Ce \implies C = e^{-1}.$$

The solution of the initial value problem is

$$y = e^{-1}xe^{x^2} = xe^{x^2-1}.$$

□

Exercise 1.3 Question 2(c)
Find the value of k so that the equation is exact and solve the equation.

$$(2xy^2 + 3x^2)dx + (2x^ky + 4y^3)dy = 0$$

Solution: We have

$$\begin{cases} M(x,y) = 2xy^2 + 3x^2 \\ N(x,y) = 2x^ky + 4y^3 \end{cases} \implies \begin{cases} \dfrac{\partial M}{\partial y} = 4xy \\ \dfrac{\partial N}{\partial x} = 2kx^{k-1}y \end{cases}.$$

The equation is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \iff 4xy = 2kx^{k-1}y \iff k = 2.$$

In this case, we set

$$\begin{aligned} F(x, y) &= \int M(x, y)\, dx \\ &= \int (2xy^2 + 3x^2)\, dx \\ &= x^2y^2 + x^3 + g(y). \end{aligned}$$

Now, we want

$$\begin{aligned} \frac{\partial F}{\partial y} &= N(x, y) \\ 2x^2y + g'(y) &= 2x^2y + 4y^3 \\ g'(y) &= 4y^3 \\ g(y) &= y^4 + C. \end{aligned}$$

The general solution of the differential equation is

$$F(x, y) = C \iff x^2y^2 + x^3 + y^4 = C.$$

□

Exercise 1.4 Question 1(e)

$$x^2y' = xy + y^2$$

Solution: Dividing both sides by x^2, we have

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Let $u = \dfrac{y}{x}$. We have

$$\begin{aligned} u + x\frac{du}{dx} &= \frac{dy}{dx} = u + u^2 \\ x\frac{du}{dx} &= u^2 \\ \frac{du}{u^2} &= \frac{dx}{x} \\ \int \frac{du}{u^2} &= \int \frac{dx}{x} \\ -u^{-1} &= \ln|x| + C \\ -\frac{x}{y} &= \ln|x| + C \\ y &= -\frac{x}{\ln|x| + C} \text{ or } y = 0. \end{aligned}$$

□

Exercise 1.5 Question 1(c)

$$xy' = y(x^2y - 1)$$

Solution: Dividing both sides by x, we have

$$\begin{aligned} y' &= xy^2 - x^{-1}y \\ y + x^{-1}y &= xy^2. \end{aligned}$$

Let $u = y^{1-2} = y^{-1}$. We have

$$\begin{aligned} \frac{du}{dx} &= (1-2)y^{-2}\frac{dy}{dx} \\ \frac{du}{dx} &= -y^{-2}(-x^{-1}y + xy^2) \\ \frac{du}{dx} &= x^{-1}y^{-1} - x \\ \frac{du}{dx} - x^{-1}u &= -x. \end{aligned}$$

An integrating factor is

$$\exp\left(\int -x^{-1}\,dx\right) = \exp(-\ln x) = x^{-1}.$$

Multiplying both sides by x^{-1}, we have

$$\begin{aligned} x^{-1}\frac{du}{dx} - x^{-2}u &= -1 \\ \frac{d}{dx}(x^{-1}u) &= -1 \\ x^{-1}u &= \int -1\,dx \\ x^{-1}u &= -x + C \\ u &= -x^2 + Cx \\ y^{-1} &= -x^2 + Cx \\ y &= (-x^2 + Cx)^{-1} \text{ or } y = 0. \end{aligned}$$

□

Exercise 1.6 Question 1(b)
Solve the following differential equation by using the given substitution.

$$y' = \sqrt{x+y}; \quad u = x + y$$

Solution: Let $u = x + y$. Then we have

$$\frac{du}{dx} = 1 + \frac{dy}{dx}.$$

Substituting it into the original differential equation, we have

$$\begin{aligned} \frac{du}{dx} - 1 &= \sqrt{u} \\ \frac{du}{dx} &= 1 + \sqrt{u} \\ \frac{du}{1+\sqrt{u}} &= dx \\ \int \frac{du}{1+\sqrt{u}} &= \int dx \\ \int \frac{du}{1+\sqrt{u}} &= x. \end{aligned}$$

Making another substitution $s = \sqrt{u}$, we have

$$ds = \frac{du}{2\sqrt{u}} \implies du = 2s\,ds,$$

and therefore

$$\begin{aligned} \int \frac{du}{1+\sqrt{u}} &= \int \frac{2s\,ds}{1+s} \\ &= 2\int \left(1 - \frac{1}{1+s}\right) ds \\ &= 2(s - \ln|1+s|) + C \\ &= 2(\sqrt{u} - \ln(1+\sqrt{u})) + C. \end{aligned}$$

Hence, we have

$$\begin{aligned} 2(\sqrt{u} - \ln(1+\sqrt{u})) &= x + C \\ 2(\sqrt{x+y} - \ln(1+\sqrt{x+y})) &= x + C \end{aligned}$$

□

Exercise 1.7 Question 1(a)

$$yy'' + (y')^2 = 0$$

Solution: Let $p = y'$. Then we have

$$y'' = p' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}.$$

Substituting it into the original differential equation, we have

$$\begin{aligned}
y\left(p\frac{dp}{dy}\right) + p^2 &= 0 \\
yp\frac{dp}{dy} &= -p^2 \\
\frac{dp}{p} &= -\frac{dy}{y} \\
\int \frac{dp}{p} &= -\int \frac{dy}{y} \\
\ln|p| &= -\ln y + C_0 \\
2C_1 p &= y^{-1}, \text{ where } C_1 = \pm\frac{1}{2}e^{-C_0} \\
2C_1\frac{dy}{dx} &= y^{-1} \\
2C_1 y\,dy &= dx \\
\int 2C_1 y\,dy &= \int dx \\
C_1 y^2 + C_2 &= x
\end{aligned}$$

□

Exercise 1.7 Question 2(b)

$$y' = \frac{x^2 + 2y}{x}$$

Solution: Rewriting the equation, we have

$$\begin{aligned}
y' &= \frac{x^2 + 2y}{x} \\
y' - \frac{2}{x}y &= x.
\end{aligned}$$

This equation is a first order linear ODE. An integrating factor is

$$\exp\left(\int -\frac{2}{x}\,dx\right) = \exp(-2\ln x) = x^{-2}.$$

Multiplying both sides with x^{-2}, we have

$$\begin{aligned}
x^{-2}y' - 2x^{-3}y &= x^{-1} \\
\frac{d}{dx}\left(x^{-2}y\right) &= x^{-1} \\
x^{-2}y &= \int x^{-1}\,dx \\
x^{-2}y &= \ln|x| + C \\
y &= x^2(\ln|x| + C).
\end{aligned}$$

□

Exercise 1.7 Question 2(d)

$$xy' + 2y = 6x^2\sqrt{y}$$

Solution: Dividing both sides by x, we have

$$y' + \frac{2}{x}y = 6x\sqrt{y}.$$

This equation is a Bernoulli's equation. Let $u = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$. We have

$$\begin{aligned}
\frac{du}{dx} &= \left(1 - \frac{1}{2}\right) y^{-\frac{1}{2}} \frac{dy}{dx} \\
\frac{du}{dx} &= \frac{1}{2}y^{-\frac{1}{2}} \left(-\frac{2}{x}y + 6x\sqrt{y}\right) \\
\frac{du}{dx} &= -\frac{1}{x}y^{\frac{1}{2}} + 3x \\
\frac{du}{dx} + \frac{1}{x}u &= 3x.
\end{aligned}$$

An integrating factor is

$$\exp\left(\int \frac{1}{x}\, dx\right) = \exp(\ln x) = x.$$

Multiplying both sides with x, we have

$$\begin{aligned}
x\frac{du}{dx} + u &= 3x^2 \\
\frac{d}{dx}(xu) &= 3x^2 \\
xu &= \int 3x^2\, dx \\
xu &= x^3 + C \\
u &= x^2 + Cx^{-1} \\
y^{\frac{1}{2}} &= x^2 + Cx^{-1} \\
y &= (x^2 + Cx^{-1})^2 \text{ or } y = 0.
\end{aligned}$$

□