MMAT 5011 Analysis II

2016-17 Term 2

Assignment 4

Due date: Mar 28, 2017

You do not have to turn in the solution of optional problems. However, you are encouraged to try all the problems.

1. State and prove the Parallogram equality for inner product space.

2. Let X be a real inner product space. Prove that for any $x, y \in X$,

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

   It is known as the Polarization identity.

3. Explain why the following functions are not inner products with counter-examples:

   (a) $\langle x, y\rangle = x_1^2y_1^2 + x_2^2y_2^2$ on $\mathbb{R}^2$;

   (b) $\langle x, y\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$ on $\mathbb{R}^2$;

   (c) $\langle x, y\rangle = x_1y_1 + ix_1y_2 + ix_2y_1 + x_2y_2$ on $\mathbb{C}^2$.

   Here, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

4. Show that the norm in l^1 is not induced from an inner product.

5. Let $T : X \to X$ be a linear operator on a complex inner product space X. Suppose $\langle T(x), x\rangle = 0$ for any $x \in X$. Show that T is the zero operator. (*Hint: Consider $y + \alpha z$ for arbritary vectors $y, z \in X$ and different $\alpha \in \mathbb{C}$.*)

6. Consider the direct sum decomposition of $M_{n\times n}(\mathbb{R}) = Sym_{n\times n} \oplus Skew_{n\times n}$, where

$$Sym_{n\times n} = \{A \in M_{n\times n}(\mathbb{R}) : A = A^t\},$$

$$Skew_{n\times n} = \{A \in M_{n\times n}(\mathbb{R}) : A = -A^t\}.$$

   (a) Express $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as $A + B$ with $A \in Sym_{3\times 3}$ and $B \in Skew_{3\times 3}$.

   (b) How about a general $M \in M_{n\times n}(\mathbb{R})$? Express the corresponding A and B in terms of M and M^t.

   (c) Verify that $\dim M_{n\times n}(\mathbb{R}) = \dim Sym_{n\times n} + \dim Skew_{n\times n}$.

7. Show that in a complex inner product space, vectors $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalar $\alpha \in \mathbb{C}$.

8. Determine whether each of the following sets is convex.

   (a) The closed unit ball $\overline{B} = \{x \in X : \|x\| \leq 1\}$ in an inner produce space X.

   (b) The subset $\{(x, y) : xy < 0\} \in \mathbb{R}^2$.

9. Let X be an inner product space and $A \subset X$ be a subspace.

   (a) Show that $A \subset (A^\perp)^\perp$.

   (b) Show that $A^\perp$ is closed in X.

   (c) Consider the subset

$$Y = \{(x_1, x_2, \ldots) : \exists N > 0 \text{ such that } x_i = 0 \ \forall i \geq N\}$$

   of l^2. What is $Y^\perp$ and $(Y^\perp)^\perp$?

10. (Optional) Show, using the standard inner product on $\mathbb{R}^2$, that the diagonals of a rhombus are perpendicular. (*Hint: Suppose $x, y \in \mathbb{R}^2$ are adjacent sides of a rhombus. What are the diagonals in terms of x and y?*)